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APPLIED THEORY OF BENDING VIBRATIONS OF A PIEZOELECTRIC BIMORPH WITH A QUADRATIC ELECTRIC POTENTIAL DISTRIBUTION

A.N. Soloviev[1,2*], V.A. Chebanenko[3], I.A. Parinov[2], P.A. Oganesyan[2]

[1]Don State Technical University, Rostov-on-Don, Russian Federation

[2]Southern Federal University, I.I. Vorovich Institute of Mathematics, Mechanics and Computer Science, Rostov-on-Don, Russian Federation

[3]Federal Research Center Southern Scientific Center of the Russian Academy of Sciences, Rostov-on-Don, Russian Federation

*e-mail: solovievarc@gmail.com

Abstract. An applied theory of cylindrical bending vibrations of a bimorph plate is developed, which takes into account the nonlinear distribution of the electric potential in piezoelectric layers. Finite-element analysis of this problem showed that such distribution arises when solving the problems of finding the resonant frequencies and modes of vibration or in the case of forced oscillations during their mechanical excitation, when the electric potentials on the electrodes are zero. The quadratic distribution of the electric potential adopted in the work showed good consistency of the results with finite-element calculations for natural oscillations and steady-state oscillations for a given potential difference when the electric potential distribution is close to linear.

Keywords: plate, cylindrical bending, electro elasticity, nonuniform potential distribution

1. Introduction

It is known that piezoelectric materials are widely used as actuators, sensors and generators in the engineering and aerospace industry for the monitoring of structures, monitoring forms, active suppression of parasitic vibrations, noise reduction, etc. Such a wide apply is achieved due to its good electromechanical properties, flexibility in the design process, ease of production and high efficiency transformation, as electric energy into mechanical energy, and in the opposite direction. When using piezoelectric materials as actuators, deformations can be controlled by changing the magnitude of the applied electrical potential. In sensors, the measurement of deformation occurs due to the measurement of the induced potential. In the field of energy storage with the help of piezoelectric materials there is a transformation of free mechanical energy present in the structures into electrical energy and its subsequent transformation into low-power devices suitable for power supply. A detailed review is given in [1-3].

Typical actuators, sensors and generators, working on a bend, represent a multilayer structure consisting of several layers with different mechanical and electrical properties. The traditional design, consisting of two piezoelectric layers glued to the substrate or to each other – is called a bimorph. More complex multilayer structures are already referred to functionally graded materials.

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Various mathematical models were proposed for modeling layered structures working as a sensor, actuator and generator. Thus, in the early works [4, 5] were presented analytical solutions three-dimensional equations of the theory of electroelasticity in static cylindrical bending and free vibrations. Nevertheless, the derivation and obtaining analytical solutions of such equations in the case of arbitrary geometry is a complex problem. Another approach is the use of models with induced deformation to simulate the response of the actuator, which were used in [6, 7]. But there the electric potential was not considered as a variable describing the state. That, in turn, did not allow to obtain related electromechanical responses, but only allowed to simulate the response of the actuator. Finite element models have been proposed in many papers, for example in [8-12]. Nevertheless, they also have their drawbacks. For example, the need for large computational power when using three-dimensional elements in problems where the thickness of one layer is much smaller than the other dimensions of the structure.

When modeling piezoelectric structures, the hypothesis of the linear distribution of the electric potential over the thickness is widely used. This means that the induced potential is considered. This is useful for modeling actuators [13] and piezoelectric generators [14]. However, in some materials with polarization in thickness, when an electric field is applied, shear strains and stresses may occur [12]. In addition, shear stresses and deformations occur in multilayer piezoelectric composites [15]. In this connection, taking into account the nonlinear part of the potential is of some interest.

The paper [16] considered a sandwich model of the third order. The authors have shown that such a model gives an additional contribution to the stiffness due to the quadratic deformation of the shear and the cubic term of the electric potential. This fact was confirmed by higher natural frequencies. A number of papers [17, 18] are devoted to the development of a related refined layer-by-layer theory for finite element analysis of multilayer functionally graded piezoelectric materials. The authors used both quadratic and cubic electric potential and took into account the longitudinal potential distribution. This allowed to take into account the shear stresses and strains. Forced and free oscillations with good convergence with analytical solutions and commercial FE packages were considered. However, no graphs of the longitudinal distribution of potential were presented. In [19], a refined bound global-local theory for finite element analysis of thick piezoelectric composites operating on the shear mode was presented. The authors used a quadratic potential distribution over the thickness. Applied theories of oscillations of multilayer piezoelectric plates, taking into account the specific distribution of the electrical potential along the thickness of the structure, were developed in [20,21]. In [22] an applied theory of oscillations of piezoelectric transducers with inhomogeneous polarization was developed.

A brief review showed that the use of the nonlinear distribution of the electric potential, along with the longitudinal distribution is of some interest in the problems of calculation of multilayer actuators, as it allows more accurate modeling of shear stresses and strains arising in such structures. Nevertheless, the behavior of the nonlinear electric potential in the vicinity of resonances is not sufficiently studied. In this connection, we have developed an applied theory of cylindrical bending of bimorph piezoelectric structures, taking into account the quadratic distribution of the potential thickness along with its longitudinal change.

2. Formulation of the problem

In this paper, the plane problem of the steady bending vibrations of a plate having an infinite width in the direction x_2 is considered. The plate consists of three layers. The outer two layers are two identical layers of piezoactive material polarized in the direction of the axis x_3. Between them is a purely elastic layer. We assume that all the functions considered are independent of the variable x_2. We choose the origin of coordinates on the middle plane.

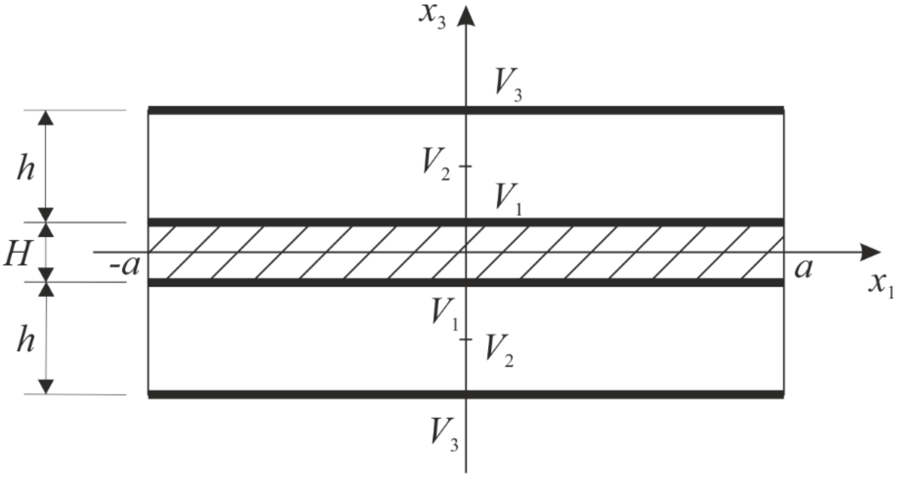

Fig. 1. The plate under study

Assume that the piezoelectric layers are deposited on the electrodes on both sides $x_3 = \pm(H/2+h)$ and $x_3 = \pm H/2$ (bold lines in Fig 1.). The external and internal electrodes are interconnected, respectively. The plate oscillations are excited by the distributed harmonic load p_i with circular frequency ω.

The oscillations of the plate are described by the following equations:

$$\sigma_{ij,j} + \rho\omega^2 u_i = p_i, \qquad D_{i,i} = 0, \tag{1}$$

where σ_{ij} - components of the stress tensor; ρ is the density of the material; u_i are the components of the displacement vector; D_i are the components of the electric induction vector. We assume that the side surface of the plate is stress-free: $\sigma_{11} = \sigma_{13} = 0$ for $x_1 = \pm a$. There are no external loads $\sigma_{13} = \sigma_{33} = 0$ on the faces $x_3 = \pm(H/2+h)$ of the plate. The external medium is air, so $D_1 = 0$ for $x_1 = \pm a$.

In this case, the constitutive relations for electroelastic medium polarized in the direction of the axis x_3 are of the form:

$$\begin{aligned} &\sigma_{11} = c_{11}^E\varepsilon_{11} + c_{13}^E\varepsilon_{33} + e_{31}\varphi_{,3}, && D_1 = 2e_{15}\varepsilon_{13} - \epsilon_{11}^S\varphi_{,1},\\ &\sigma_{33} = c_{13}^E\varepsilon_{11} + c_{33}^E\varepsilon_{33} + e_{33}\varphi_{,3}, && D_3 = e_{31}\varepsilon_{11} + e_{33}\varepsilon_{33} - \epsilon_{33}^S\varphi_{,3},\\ &\sigma_{13} = 2c_{44}^E\varepsilon_{13} + e_{31}\varphi_{,1}, && \end{aligned} \tag{2}$$

where c_{ij}^E are the elastic moduli measured with a constant electric field, ε_{ij} are the strain tensor components, e_{ij} is the piezoelectric constant, φ is the electric potential, and ϵ_{ij}^S is the permittivity measured at constant deformations.

For a purely elastic inner layer, the constitutive relations have the following form:

$$\begin{aligned} \hat{\sigma}_{11} &= c_{11}\varepsilon_{11} + c_{13}\varepsilon_{33},\\ \hat{\sigma}_{33} &= c_{13}\varepsilon_{11} + c_{33}\varepsilon_{33},\\ \hat{\sigma}_{13} &= 2c_{44}\varepsilon_{13}. \end{aligned} \tag{3}$$

Further, to construct an applied theory of oscillations, we adopt the Kirchhoff hypotheses. In accordance with them, the distribution of displacements along the thickness has the following form

$$\begin{aligned} u_1(x_1, x_3) &= -x_3 w_{,1},\\ u_3(x_1, x_3) &= w(x_1), \end{aligned} \tag{4}$$

where $w(x_1)$ is the deflection function of the middle surface of the plate.

In addition, the hypotheses assumed suggest that the normal stress is equal $\sigma_{33} = 0$ everywhere in the plate region. Using this condition, we exclude the deformation ε_{33} from the constitutive relations for the electric (2) and elastic (3) media:

$$\begin{aligned}&\sigma_{11} = c_{11}^{*} u_{1,1} + e_{31}^{*} \varphi_{,3},\\ &D_3 = e_{31}^{*} u_{1,1} - \epsilon_{33}^{*} \varphi_{,3},\\ &\hat{\sigma}_{11} = \tilde{c}_{11}^{*} u_{1,1},\end{aligned} \tag{5}$$

where

$$\begin{aligned}&c_{11}^{*} = c_{11}^{E} - \frac{c_{13}^{E\,2}}{c_{33}^{E}}, \qquad e_{31}^{*} = e_{31} - \frac{c_{13}^{E} e_{33}}{c_{33}^{E}},\\ &\epsilon_{33}^{*} = \epsilon_{33}^{S} + \frac{e_{33}^{2}}{c_{33}^{E}}, \qquad \tilde{c}_{11} = c_{11} - \frac{c_{13}^{2}}{c_{33}}.\end{aligned} \tag{6}$$

Expressions for $\sigma_{13}, \hat{\sigma}_{13}$ and D_1 remain unchanged.

We assume that the electric potential for the upper piezoelectric layer has the following distribution:

$$\varphi(x_1, \tilde{x}_3) = V_1(x_1) \frac{\tilde{x}_3}{h} \left(\frac{2\tilde{x}_3}{h} - 1 \right) + V_2(x_1) \left(1 - \frac{4\tilde{x}_3^2}{h^2} \right) + V_3(x_1) \frac{\tilde{x}_3}{h} \left(\frac{2\tilde{x}_3}{h} + 1 \right). \tag{7}$$

Here, for the convenience of the description, the relative coordinate $\tilde{x}_3 = x_3 - (H/2 + h/2)$ is introduced. In the lower layer we assume an analogous distribution for $\tilde{x}_3 = x_3 + (H/2 + h/2)$.

Using the electric potential in the form (7) allows to take into account the electric boundary conditions on $x_3 = \pm(H/2 + h)$ and $x_3 = \pm H/2$, as well as the value in the middle of the piezoactive layers $x_3 = \pm(H/2 + h/2)$. In the framework of the problem under study, let us consider the following case:

$$\begin{aligned}&V_1(x_1) = V_1 = const,\\ &V_2(x_1) = \Phi(x_1),\\ &V_3(x_1) = V_3 = const.\end{aligned} \tag{8}$$

Here $\Phi(x_1)$ is the unknown distribution function of the potential in the middle of the piezoactive layer in the direction of the x_1 axis.

Next, we use the variational equation for the case of steady oscillations, which generalizes the Hamilton principle in the theory of electroelasticity. For the case of plane deformation in the absence of surface loads and surface charges, the variational equation has the form:

$$\int_{-a}^{a} \int_{-h}^{h} \delta \breve{H} dx_3 dx_1 - \rho\omega^2 \int_{-a}^{a} \int_{-h}^{h} u_i \delta u_i dx_3 dx_1 + \int_{-a}^{a} \int_{-h}^{h} p_i \delta u_i dx_3 dx_1 = 0, \tag{9}$$

where $\breve{H} = U - E_i D_i$ is the electric enthalpy whose variation is equal to $\delta\breve{H} = \sigma_{ij}\delta\varepsilon_{ij} - D_i \delta E_i$.

Taking into account the accepted hypotheses (4), the enthalpy variation takes the following form:

$$\delta\breve{H} = \sigma_{11}\delta\varepsilon_{11} - D_1\delta E_1 - D_3\delta E_3. \tag{10}$$

We assume that the components of the vector of distributed load are $\mathbf{p} = \{0, p\}^T$. We vary (10) and substitute it in (9). After integration over the thickness, we equate the coefficients for independent variations of δw and $\delta\Phi$. Thus, we obtain a system of differential equations

$$\begin{gathered}\frac{4}{3}e_{31}^{*}h\frac{d^2\Phi}{dx_1^2}+\left(\frac{1}{12}\tilde{c}_{11}^{*}H^3+\frac{1}{2}c_{11}^{*}H^2h+c_{11}^{*}Hh^2+\frac{2}{3}c_{11}^{*}h^3\right)\frac{d^4w}{dx_1^4}\\+\left(\frac{1}{12}\tilde{\rho}\omega^2H^3+\frac{1}{2}\rho\omega^2H^2h+\rho\omega^2Hh^2+\frac{2}{3}\rho\omega^2h^3\right)\frac{d^2w}{dx_1^2}\\-\left(H\omega^2\tilde{\rho}+2h\omega^2\rho\right)w-p=0,\\\frac{16}{15}h\epsilon_{11}^{S}\frac{d^2\Phi}{dx_1^2}+\frac{4}{3}he_{31}^{*}\frac{d^2w}{dx_1^2}-\frac{32}{3}\frac{\epsilon_{33}^{*}}{h}\Phi+\frac{16}{3}\frac{\epsilon_{33}^{*}}{h}V_1+\frac{16}{3}\frac{\epsilon_{33}^{*}}{h}V_3=0.\end{gathered}\tag{11}$$

Equating the coefficients of independent variations of the nonintegral terms to zero, we obtain the boundary conditions:

$$\begin{gathered}\frac{4}{3}e_{31}^{*}h\frac{d\Phi}{dx_1}+\left(\frac{1}{12}\tilde{c}_{11}^{*}H^3+\frac{1}{2}c_{11}^{*}H^2h+c_{11}^{*}Hh^2+\frac{2}{3}c_{11}^{*}h^3\right)\frac{d^3w}{dx_1^3}\\+\left(\frac{1}{12}\tilde{\rho}\omega^2H^3+\frac{1}{2}\rho\omega^2H^2h+\rho\omega^2Hh^2+\frac{2}{3}\rho\omega^2h^3\right)\frac{dw}{dx_1}=0,\\-\left(e_{31}^{*}H+\frac{5}{3}e_{31}^{*}h\right)V_3+\left(e_{31}^{*}H+\frac{1}{3}e_{31}^{*}h\right)V_1+\frac{4}{3}e_{31}^{*}h\Phi\\+\left(\frac{1}{12}\tilde{c}_{11}^{*}H^3+\frac{1}{2}c_{11}^{*}H^2h+c_{11}^{*}Hh^2+\frac{2}{3}c_{11}^{*}h^3\right)\frac{d^2w}{dx_1^2}=0,\\-\frac{16}{15}h\epsilon_{11}^{S}\frac{d\Phi}{dx_1}=0.\end{gathered}\tag{12}$$

3. Numerical experiment

Using the obtained model, we investigate a plate made of piezoceramics PZT-4 fixed with hinges at points $x_1=\pm a$. The inner layer is made of the same material, but does not have piezoactive properties. In view of the foregoing, the basic physical and geometric parameters of the model were given in the table.

Table 1. Geometrical parameters and physical properties

Parameter	Value	Dimension
Linear dimensions	$H=2\times10^{-3}$, $h=5\times10^{-3}$, $a=0.1$,	m
Density	$\rho=\tilde{\rho}=7.5\times10^{3}$	kg/m^3
Modules of elasticity	$c_{11}^{E}=c_{11}=13.9\times10^{10}$, $c_{13}^{E}=c_{13}=7.43\times10^{10}$, $c_{33}^{E}=c_{33}=11.5\times10^{10}$	GPa
Piezoelectric modules	$e_{15}=12.7$, $e_{31}=-5.2$, $e_{33}=15.1$,	C/m^2
Permittivity	$\epsilon_{11}^{S}=64.6\times10^{-10}$, $\epsilon_{33}^{S}=56.2\times10^{-10}$	F/m

We will compare the results of the proposed model with the results of the finite element (FE) analysis of a similar problem in the FE package ACELAN [23].

At the first stage, we find the first two modes of oscillation, under the condition $V_1=V_3=0$:

Table 2. Resonance frequencies

Mode of oscillation	Applied theory (Hz)	FE (Hz)	Error (%)
First	473.8	481.1	1.51
Second	1895.3	1881.8	0.71

Comparison of the results between applied theory and FE modelling showed a small spread between the results obtained

Next, consider the oscillations of the plate at a frequency of 1890 Hz, with the condition $V_1 = V_3 = 0$ and $p = 1$.

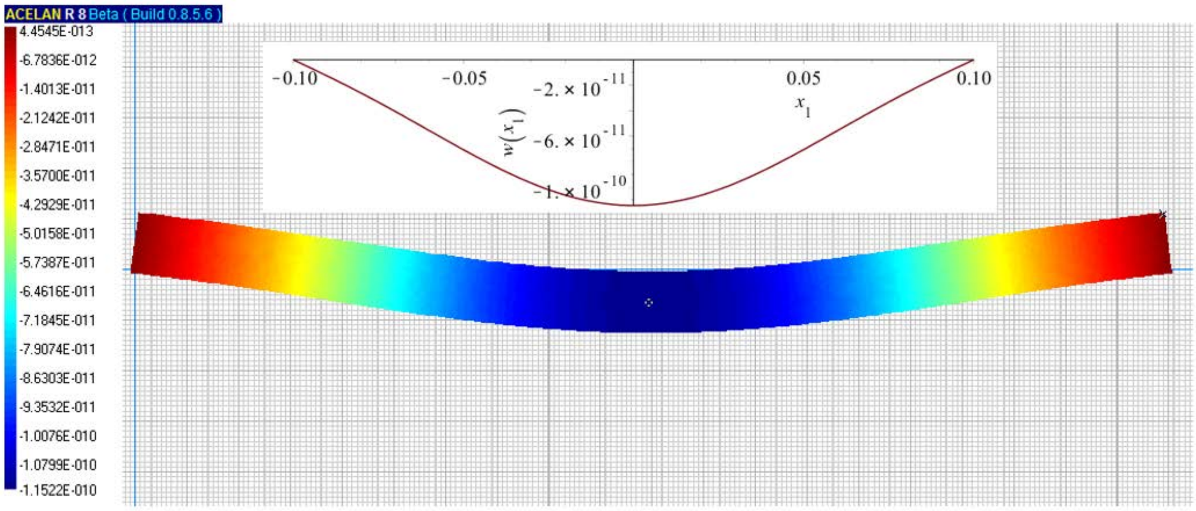

Fig. 2. Deflection of the plate obtained on the basis of applied theory (plot in the upper part of figure) and FE method

Figure 2 demonstrates a good agreement between applied theory and finite element calculation.

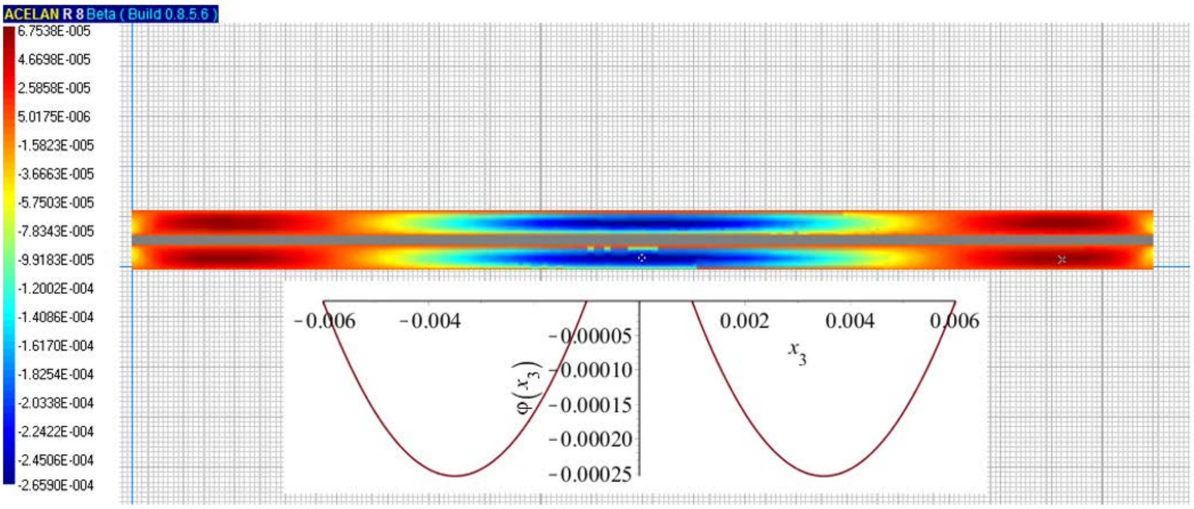

Fig. 3. Electrical potential distribution though the thickness for the middle of the plate, obtained on the basis of the applied theory (plot in the lower part of figure), and for the whole plate, obtained by the FE method

The analysis of Fig. 3 demonstrates the nonlinear character of the distribution of the electrical potential along the thickness and length of the piezoactive layer, as well as the similarity of the results of applied theory and finite element analysis.

Figure 4 illustrates the distribution of the electrical potential along the length and thickness of the upper layer. Near the plate fixing points, local maxima of the electric potential values are observed, and in the middle - a minimum.

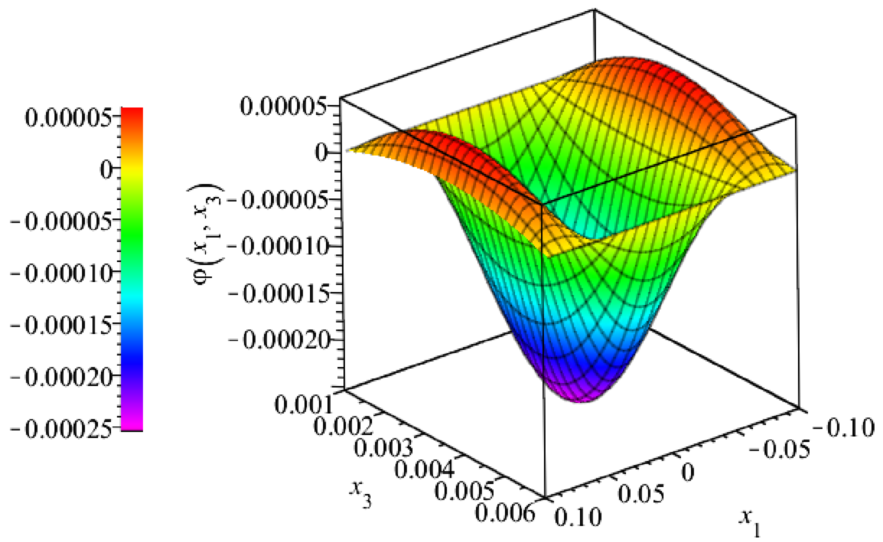

Fig. 4. Electrical potential distribution for the upper piezoactive layer, obtained on the basis of applied theory

Next, consider the case when the electric potential is $V = -7.65\,\mathrm{V}$ on internal electrodes, and the potential on external electrodes is $V = 0$. The oscillations are excited by the action of a distributed force with an amplitude of 1000 N and a frequency of 1890 Hz.

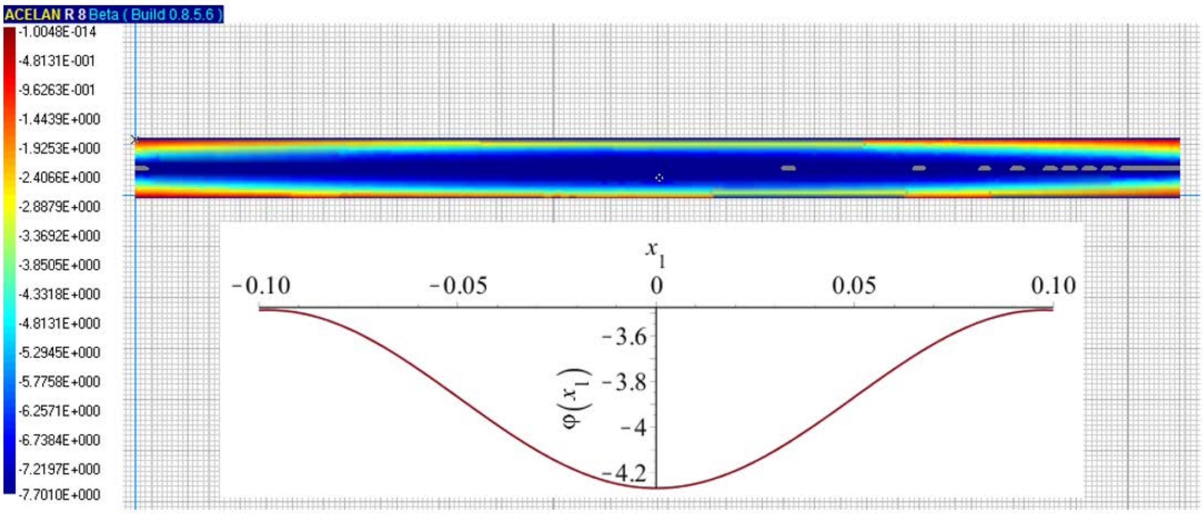

Fig. 5. Distribution of the electrical potential along the length in the middle of the upper piezoactive layer, obtained on the basis of the applied theory (plot in the lower part of figure), and for the entire plate, obtained by the FE method

It can be seen from Fig. 5 that the values of distribution the electric potential, obtained on the basis of applied theory, are rather close to those obtained on the basis of finite element analysis. In addition, the distribution has a nonlinear form.

Figure 6 shows the distribution of the electrical potential along the length and thickness of the upper piezoactive layer.

Analysis of Fig. 5 and 6 allows us to conclude that in the case when an electric potential different from zero is specified on one of the electrodes, the form of the electric potential distribution along the thickness is close to linear. However, the distribution of the electrical potential along the length of the piezoceramic layer is nonlinear, with a difference of 22% in the middle of the plate.

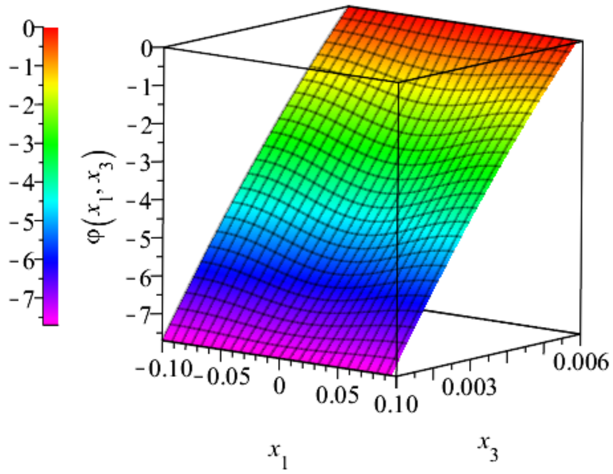

Fig. 6. Distribution of electrical potential for the upper piezoactive layer, obtained on the basis of applied theory

4. Conclusions

An applied theory of oscillations of a bimorph plate is developed, which takes into account the nonlinear distribution of the electric potential in piezoelectric layers. Such a distribution arises when solving the problems of finding the resonant frequencies and modes of vibration or in the case of forced oscillations during their mechanical excitation, when the electric potentials on the electrodes are zero. The quadratic distribution of the electric potential adopted in the work showed good consistency of the results with finite-element calculations for natural oscillations and steady-state oscillations for a given potential difference when the electric potential distribution is close to linear

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