

# QUANTUM QUARTIC OSCILLATORS WITH TWO COUPLED DEGREES OF FREEDOM

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**Abstract.** The numerical integration of the non-stationary two-dimensional Schrödinger equation is carried out. The dynamical properties (dynamical averages, frequency spectra, uncertainty relations, autocorrelators) for two quantum oscillators coupled by the Pallen-Edmonds potential are studied. As a result of the numerical simulations, it was established that the quantum system is sensitive to small changes in Hamiltonian that are caused by the Pallen-Edmonds coupling potential. In the regime of weak coupling high-frequency oscillations are generated, the spectral component number is increased at amplification of coupling.

## 1. Introduction

Quantum oscillators with two coupled degrees of freedom have been the subject of numerous investigations. At the present time, the quantum harmonic oscillators with the Pallen-Edmonds coupling potential [1] are sufficiently well investigated. The given potential is defined as the product of positions (Cartesian coordinates) rose to the second power and a constant parameter. The parameter introduces the coupling between the degrees of freedom. In accordance with the parameter, there are possible the distinct oscillatory regimes that can be caused by the behavior of the classical analog of a quantum oscillator. In this case we speak about the quantum-classical correspondence. Greater importance for development of theory of the quantum informative processes is a quantum anharmonic oscillator with the coupling between the degrees of freedom. That's why the investigation of quantum dynamics of such systems is a modern problem.

Previously, the quantum oscillators with two degrees of freedom and the quadratic potential including the Pallen-Edmonds potential have been investigated in classic, semi classic and quantum description [2, 3], where particular attention has been given to the quantum oscillators, which are chaotic in classic limit. In the last few years an extensive literature has evolved in this direction [4-7]. In paper [7], a special case of a quantum quartic oscillator is discussed. Here, the non-dimensional polynomial potential is taken in the form  $3x^4 + y^4 + \gamma x^2 y^2$ , the coupling parameter  $\gamma$  being varied in a certain interval. The classical analog of such system in chaotic regime and its influence on quantum oscillator was also investigated.

In parallel with the dynamical chaos there is also another problem of the quantum quadratic oscillator, when the broad-band discrete frequency spectra are generated in classical limit and the dynamical chaos is absent. The present paper is devoted to this topic. We studied the influence of the Pallen-Edmonds coupling potential on the frequency spectra evolution. We continue the series of our works [8-11].

## 2. Basic equations and assumptions

The Schrödinger equation written in the dimensionless form

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi + U_{\Sigma} \psi \quad (1)$$

defines the time evolution of the wave function  $\psi = \psi(x, y, t)$ ;  $x, y, t$  –are the coordinates and time, respectively. Equation (1) is invariant to the choice of measurement units. The total potential  $U_{\Sigma}$  is determined as

$$U_{\Sigma} = U_1(x, y) + U_c(x, y), \quad (2)$$

where the terms are polynomial

$$U(x, y) = a_x x^4 + a_y y^4$$

$$U_c(x, y) = \gamma \cdot x^2 y^2. \quad (3)$$

The coefficients  $a_x, a_y$  and the coupling parameter  $\gamma$  are variable quantities. Below, we discuss the solutions of the Schrödinger equation at  $a_x = a_y$  and the Pallen-Edmonds potential  $U_c$  under small values  $\gamma$ . The quantum system investigated is located between the walls of the infinite potential well; it is spatially confined. The coordinates of the well are  $\pm x_{\alpha}, \pm y_{\alpha}$ , and  $x_{\alpha} = y_{\alpha}$ . In these points the values  $U, U_c$  are finite. The influence of the walls on oscillations depends on the well dimensions and the initial energy of quantum wave packets. In the whole, we have the model of the plane cell. If  $U_c = 0$ , the dynamical system can be represented as consisting of two independent oscillators. But, if  $U_c \neq 0$ , the system becomes coupled.

The boundary conditions on the external walls of the quantum system and the initial wave function are specified as

$$\psi(\pm x_{\alpha}, \pm y_{\alpha}, t) = 0, \quad \psi(x, y, t=0) = \psi_0(x, y). \quad (4)$$

The initial condition is specified in the form of Gaussian wave packet

$$\psi_0(x, y) = C \exp \left[ -\alpha_x (x - x_0)^2 - \alpha_y (y - y_0)^2 \right] \quad (5)$$

where  $\alpha_x = \alpha_y$ ,  $x_0, y_0$  are the mean coordinates of the quantum wave packet at  $t=0$ . Constant  $C$  is defined numerically from the normalization condition.

The dynamical properties of the quantum wave packets are studied via mean positions  $\langle x \rangle, \langle y \rangle$ , and mean velocities  $\langle V_x \rangle, \langle V_y \rangle$ . The normal deviations  $\langle \Delta x \rangle, \langle \Delta y \rangle$ ,  $\langle \Delta V_x \rangle, \langle \Delta V_y \rangle$  in the uncertainty relations

$$\Delta x \Delta V_x \geq \frac{1}{2}, \quad \Delta y \Delta V_y \geq \frac{1}{2}. \quad (6)$$

are designated as  $g_x, g_y$ , respectively. We investigated time evolution of the above listed quantities in detail. In addition to the time realizations, the frequency spectra of the dynamical variables were also studied. The Fourier transformation for  $\langle x \rangle, \langle y \rangle$  was performed and

the corresponding amplitudes were designated as  $F_{\langle x \rangle}(\Omega)$ ,  $F_{\langle y \rangle}(\Omega)$ , the quantity  $\Omega$  is a frequency. The wave packet autocorrelation function as supplementary means in the identification of dynamical properties was calculated. It has the following form

$$R = \left| \int_{-x_\alpha}^{x_\alpha} dx \int_{-y_\alpha}^{y_\alpha} dy \psi^*(x, y, t) \psi_0(x, y) \right|^2. \quad (7)$$

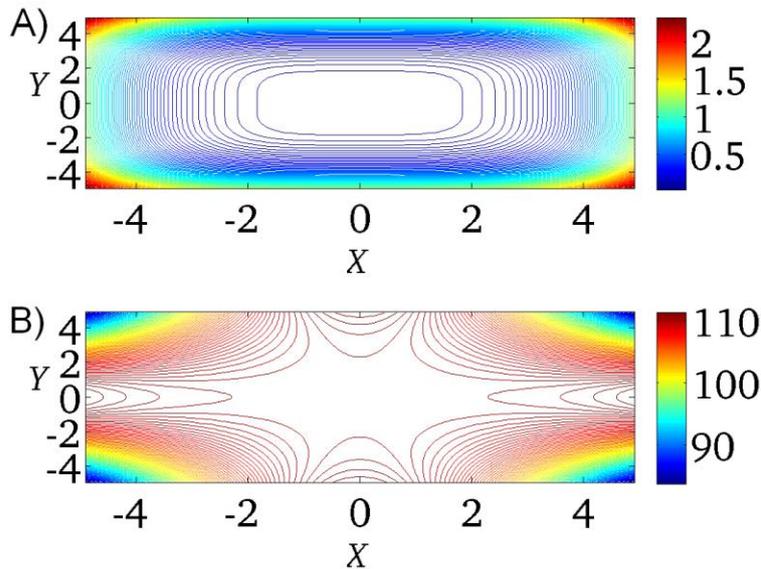
The numerical integration of the non-stationary two-dimensional Schrödinger equation (1) was carried out in a closed region

$$\bar{R} = \{(x, y, t) : -7 \leq x \leq 7, -7 \leq y \leq 7, 0 \leq t \leq 1500\}.$$

Alternatively, if the time interval  $T > 1500$ , the minimal frequency  $\Omega_{\min} = \frac{2\pi}{T}$  becomes less and the definition precision of the frequencies increases.

### 3. Time evolution and frequency response under the weak coupling

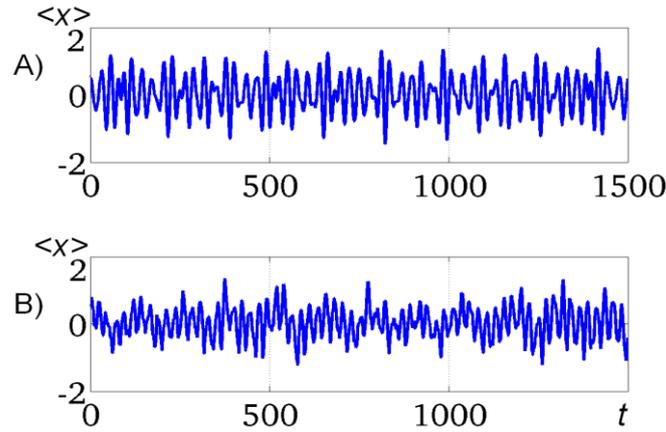
Contrary to the investigations performed in [3, 5], we study the quantum dynamical system with the same parameters  $a_x = a_y = 0.002$  and small coupling  $\gamma \in [0; 0.05]$ . The initial Gaussian packet is characterized by parameters  $\alpha_x = \alpha_y = 0.5$ , and the mean positions  $x_0 = y_0 = 0.5$ . The mean positions are much less than the half size of the well, which is equal to 7. The isoline map of the potential at different values of  $\gamma$  is presented in Fig. 1. At  $\gamma = 0$  the potential isolines have a simple relief. The numerical calculations of oscillatory motions were carried out at  $\gamma = 0; 0.001; 0.005; 0.05$ . Under the small variations of  $\gamma$  the relief is deformed slightly, but at  $\gamma = 0.05$  dissimilarity from that  $\gamma = 0$  becomes evident. The time realizations of  $\langle x \rangle$ , corresponding to maps are presented in Fig. 2.



**Fig. 1.** Isoline map of the potential at various  $\gamma$ : A)  $\gamma = 0$ , B)  $\gamma = 0.05$ .

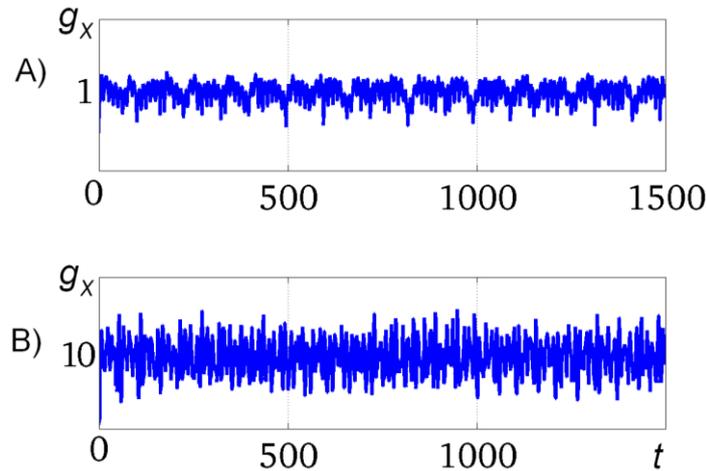
The analogous solutions take place for time realization of the mean position  $\langle y \rangle$ , the evolution pictures  $\langle x \rangle$ ,  $\langle y \rangle$  coincide. Influence of the parameter  $\gamma$  on the evolution of

the wave packets is conveniently to discuss in the context of changing for  $g_x$  and frequency spectra. In connection with the potential symmetry, all calculations at the specified parameters bring into coincidence for both degrees of freedom  $g_x = g_y$ ,  $F_{\langle x \rangle}(\Omega) = F_{\langle y \rangle}(\Omega)$ .



**Fig. 2.** Time realizations of mean  $\langle x \rangle$  at different  $\gamma$ ; A)  $\gamma = 0$ , B)  $\gamma = 0.05$ .

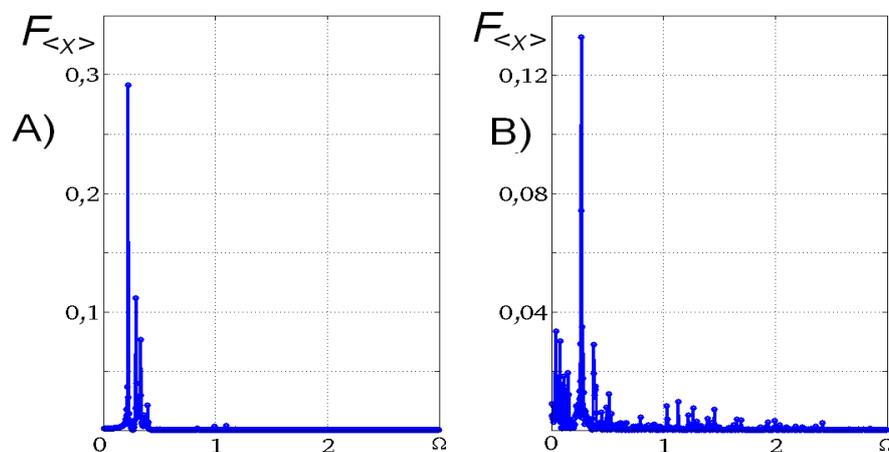
If  $\gamma = 0$ , the uncertainty product  $g_x$  oscillates about  $g_x \approx 1$ . At  $\gamma = 0.001$  it exceeds scarcely this level. But at  $\gamma = 0.005$  the situation is essentially changed. Now, the oscillations occur about the value  $g_x \approx 2$ . Finally, at  $\gamma = 0.05$  the product  $g_x$  oscillates about the value  $g_x \approx 10$  (Fig. 3).



**Fig. 3.** Influence of coupling parameter  $\gamma$  on the product  $g_x$ ; A)  $\gamma = 0$ , B)  $\gamma = 0.05$ .

More dramatic changes of frequency spectra take place with increasing the coupling parameter  $\gamma$ . In Fig. 4 the frequency spectra are presented for different values of  $\gamma$ . The noticeable changes occur at  $\gamma = 0.005$  in comparison with oscillatory regime at  $\gamma = 0$ . Firstly, the amplitude of the most intensive spectral components decreases and new components are generated gradually. In case of further increase of  $\gamma$ , the number of new spectral components is growing rapidly and the energy redistribution of spectra is going on simultaneously. In separate cases the time interval  $T$  was increased, and as a result of which the value of  $\Omega_{\min}$  was decreased. It was done in order to discriminate the closely-spaced spectral components  $F_{\langle x \rangle}(\Omega)$ ,  $F_{\langle x \rangle}(\Omega + \Delta\Omega)$  and to fulfill the condition  $\Delta\Omega > \Omega_{\min}$ .

The choice and calculations of spectral components number for the same time interval  $T$  were performed in such a manner that the components with the amplitude less than specified minimal value were ignored. In Fig. 4 we can observe the growing of spectral component number including that at  $\Omega > 1$ . Two more intensive spectral components vary with  $\gamma$  as shown in Table 1.

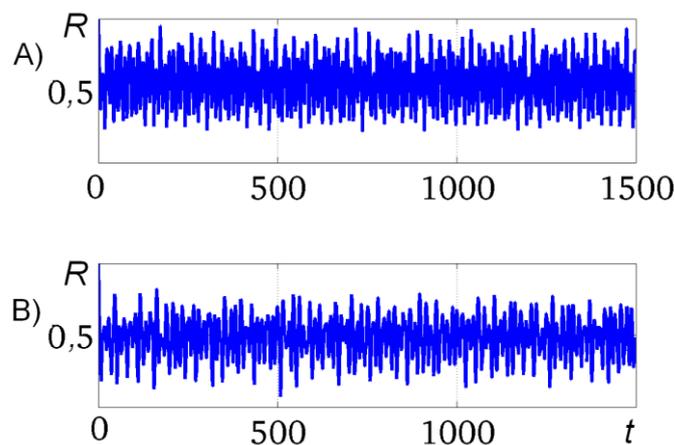


**Fig. 4.** Frequency spectra at different  $\gamma$ ; A)  $\gamma = 0$ , B)  $\gamma = 0.05$ .

Table 1. Spectral components  $F_{\langle x \rangle}(\Omega)$ ,  $F_{\langle y \rangle}(\Omega)$  at different  $\gamma$ .

$\gamma$	0	0.001	0.005	0.05
$\Omega$	0.2178	0.2094	0.1424	0.2681
$F_{\langle x \rangle}(\Omega)$	0.2912	0.2010	0.1777	0.1328
$\Omega$	0.2898	0.1968	0.2053	0.2765
$F_{\langle y \rangle}(\Omega)$	0.1117	0.0661	0.1555	0.0350

The calculations of autocorrelators at different  $\gamma$  are illustrated in Fig. 5. They amplify the picture of multi-frequency process corresponding to the discrete spectrum. The investigated two-dimensional oscillations on the plane  $(x, y)$  are synchronous. This property can be described by the function  $\langle y \rangle = f(\langle x \rangle)$ , where  $f(\langle x \rangle)$  is varied along the bisectrix of a right corner.



**Fig. 5.** Autocorrelators as function of time at different  $\gamma$ ; A)  $\gamma = 0$ , B)  $\gamma = 0.05$ .

#### 4. Conclusions

In the context of numerical solution of the non-stationary two-dimensional Schrödinger equation, the quartic oscillators were investigated. The analysis of quantum quartic oscillator with two degrees of freedom was performed during which the time realizations for mean positions and their frequency spectra, uncertainty products, autocorrelators were studied in details. As a result of the numerical simulations, it was established that the quantum system is sensitive to small changes in Hamiltonian that are caused by the Pallen-Edmonds coupling potential. Under increasing a coupling parameter, the spectral component number is increased markedly and the frequency spectra of mean positions are displaced to lower and higher frequencies. The broad-band properties are observed even at the weak coupling between two degrees of freedom. Simultaneously, the phase volume as the measure of the quantum fluctuations is increased also when the coupling parameter is growing. The given investigations represent not only the general theoretical interest, but it can be used for the description of mesoscopical systems and other quantum objects.

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