DYNAMIC RESPONSE OF A HEAT CONDUCTING SOLID BAR OF POLYGONAL CROSS SECTIONS SUBJECTED TO MOVING HEAT SOURCE

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Abstract. The dynamic response of a heat conducting solid bar of polygonal cross section subjected to moving heat source is discussed using the Fourier expansion collocation method (FECM). The equations of motion are formulated using the three dimensional constitutive equation of elasticity and generalized thermo elastic equation composed of linear homogeneous isotropic material. Three displacement potential functions are introduced to uncouple the equations of motion and the heat conduction. The frequency equations are obtained by satisfying the boundary conditions along the surface of the polygonal solid bar using Fourier expansion collocation method. The numerical calculations are carried out for triangular, square, pentagonal and hexagonal cross sectional bars with different moving heat source speeds. Dispersion curves are plotted for longitudinal and flexural (antisymmetric) modes of non dimensional frequency.

1. Introduction

The study of dynamic response of heat conducting solid bar is significant in the ultrasonic inspection of materials, vibration of engineering structures, atomic physics, industrial engineering, thermal power plants, submarine structures, pressure vessel, aerospace, chemical pipes and metallurgical process. The theory of moving sources of heat has been instrumental in providing the welding engineer with a scientific criterion for the weld ability of steel and surface hardening of metallic alloys. The importance of thermal stresses in causing structural damages and changes in functioning of the structure is well recognized whenever thermal stress environments are involved. Therefore, the ability to predict thermodynamics stresses induced by moving heat sources in structures with polygonal cross sections are essential for the proper and safe design and the knowledge of its response during the service in these severe thermal environments. The dispersion of displacement, temperature change and perturbed magnetic field in case of fundamental modes for the symmetric and antisymmetric cases of the cylindrical panel is playing a vital role in smart material applications. This type of model analysis is very important in bio sensing applications in nuclear magnetic resonance (NMR), magnetic resonance imaging (MRI) and echo planar imaging (EPI).

Gazis [1] has studied the most general form of harmonic waves in hollow cylinder of circular cross section of infinite length. He presented in detail the frequency equation in Part I and numerical results in Part II. Mirsky [2] analyzed the wave propagation in transversely isotropic circular cylinders of infinite length and presented the frequency equation in Part I and numerical results in Part II. Nagaya [3-6] discussed wave propagation in an infinite bar of arbitrary cross section and the wave propagation in an infinite cylinder of both inner and outer

arbitrary cross section applicable to a bar of general cross section, based on three-dimensional theory of elasticity. The boundary conditions along the free surface of arbitrary cross section are satisfied by means of Fourier expansion collocation method. Paul and Venkatesan [7] have studied the wave propagation in an infinite piezoelectric solid cylinder of arbitrary cross section using Fourier expansion collocation method.

Ashida [8, 9] have presented the temperature and stress analysis of an elastic circular cylinder in contact with heated rigid stamps and thermally - induced wave propagation in a piezoelectric plate respectively. Tauchert et al. [10] discussed the developments in thermo piezoelectricity with relevance to smart composite structures. The equations governing linear response of piezoelectric media are outlined, and general solution procedure based on potential functions was described. Gao and Noda [11] have studied the thermal-induced interfacial cracking of magnetoelectroelastic materials under uniform heat flow. Chen et al. [12] analyzed the point temperature solution for a pennay-shapped crack in an infinite transversely isotropic thermo-piezo-elastic medium subjected to a concentrated thermal load applied arbitrarily at the crack surface using the generalized potential theory. Suhubi [13] studied the longitudinal vibration of a circular cylinder coupled with a thermal field. The frequency equation was obtained for two particular cases, namely for a constant temperature and for zero heat flux on the surface of the cylinder and solved the frequency equation for small radii and weak coupling. Later with Erbay [14], he studied the longitudinal wave propagation in a generalized thermoelastic infinite cylinder and obtained the dispersion relation for a constant surface temperature of the cylinder. Lord and Shulman [15] formulated a generalized dynamical theory of thermoelasticity using the heat transport equation that included the time needed for the acceleration of heat flow.

Green and Lindsay [16] presented an alternative generalization of classical thermoelasticity. Restrictions on constitutive equations were discussed with the help of an entropy production inequality proposed by Green and Laws [17]. They have showed that the linear heat conduction tensor was symmetric and that the theory allows for second sound effects. Hallam and Ollerton [18] investigated the thermal stresses and deflections that occurred in a composite cylinder due to a uniform rise in temperature, experimentally and theoretically and compared the obtained results by a special application of the frozen stress technique of photoelasticity. Singh and Sharma [19] investigated the generalized thermoelastic waves in transversely isotropic media. The wave propagation of plane harmonic waves in anisotropic generalized thermoelasticity was developed by Sharma and Sidhu [20]. Sharma [21] discussed the three-dimensional vibration analysis of a homogeneous transversely isotropic thermoelastic cylindrical panel.

Varma [22] presented the propagation of waves in layered anisotropic media in generalized thermo elasticity in a arbitrary layered plate. The dispersion relations of thermoelastic waves were obtained by invoking continuity at the interface and boundary conditions on the surfaces of layered plate. Savoia and Reddy [23] studied the three dimensional thermal analysis of laminated composite plates subjected to thermal and mechanical loads in the context of the three dimensional quasi-state theory of thermoelasticity.

Selvamani [24] has obtained the frequency equation of the flexural wave motion of a heat conducting doubly connected thermo elastic plate of polygonal cross sections using Fourier expansion collocation method. The frequency equations are obtained for longitudinal and flexural vibrations and are studied numerically for different cross section sectional plates. Recently, Ponnusamy and Selvamani [25, 26] have studied respectively, the three dimensional wave propagation of a transversely isotropic magneto thermo elastic and generalized thermo elastic cylindrical panel in the context of the linear theory of thermo elasticity.

The problems involving a moving heat source plays vital role in thermal fields due to its extensive engineering applications, such as continuous annealing after cold working, pulsedlaser cutting and welding, and high speed machining and grinding etc. Al-Huniti et al. [27] studied the dynamic responses of a copper rod due to a moving heat source under the wave type heat conduction model and by means of the Laplace transform the temperature was obtained directly from the heat conduction equation. Baksi et al. [28] considered a threedimensional problem for a homogeneous, orthotropic, electrically as well as thermally conducting infinite rotating elastic medium with heat source by the eigen value approach. three-dimensional Lykotrafitis and Georgiadis [29] studied the steady-state thermo elastodynamic problem of moving sources over a half space. Hsieh [30] presented the exact solution of Stefan problems related to a moving line heat source in a quasi-stationary state.

In this paper, the free vibration of a generalized thermoelastic solid cylinder of polygonal (triangular, square, pentagonal and hexagonal) cross section is studied using the Fourier expansion collocation method based on Sububi's generalized theory [31]. The computed nondimensional wave numbers are plotted as graphs.

2. Governing equations

The generalized theories of thermoelasticity are the constitutive equations in which the functions are temperature rate dependent. The constitutive equations for a linear isotropic thermoelastic medium, the stresses σ_{ii} , expressed in cylindrical coordinates, are

$$\sigma_{rr} = \lambda \left(e_{rr} + e_{\theta\theta} + e_{zz} \right) + 2\mu e_{rr} - \beta \left(T + \eta T_{,t} \right), \tag{1a}$$

$$\sigma_{\theta\theta} = \lambda \left(e_{rr} + e_{\theta\theta} + e_{zz} \right) + 2\mu e_{\theta\theta} - \beta \left(T + \eta T_{t} \right), \tag{1b}$$

$$\sigma_{zz} = \lambda \left(e_{rr} + e_{\theta\theta} + e_{zz} \right) + 2\mu e_{zz} - \beta \left(T + \eta T_{,t} \right), \tag{1c}$$

$$\sigma_{r\theta} = 2\mu\gamma_{r\theta}, \quad \sigma_{\theta z} = 2\mu\gamma_{\theta z}, \quad \sigma_{rz} = 2\mu\gamma_{rz}.$$
(1d,e,f)

where e_{ij} are the strain components, β is the thermal stress coefficients, T is the temperature, η is of generalized thermo elasticity, t is the time, λ and μ are Lame' constants. The strain e_{ij} are related to the displacements are given by

$$e_{rr} = u_{,r}, \quad e_{\theta\theta} = r^{-1} \left(u + v_{,\theta} \right), \quad e_{zz} = w_{,z},$$
 (2a)

$$\gamma_{r\theta} = v_{,r} - r^{-1} (v - u_{,\theta}), \qquad \gamma_{z\theta} = v_{,z} + r^{-1} w_{,\theta}, \qquad \gamma_{rz} = w_{,r} + u_{,z},$$
 (2b)

in which u, v and w are the displacement components along radial, circumferential and axial directions respectively. The comma in the subscripts denotes the partial differentiation with respect to the variables.

3. Equations of motion

The three dimensional equations of motion and the heat conduction equation in the reference system r, θ and z are

$$\sigma_{rr,r} + r^{-1}\sigma_{r\theta,\theta} + \sigma_{rz,z} + r^{-1}(\sigma_{rr} - \sigma_{\theta\theta}) = \rho u_{,tt}, \qquad (3a)$$

$$\sigma_{r\theta,r} + r^{-1}\sigma_{\theta\theta,\theta} + \sigma_{rzz} + \sigma_{\theta z,z} + 2r^{-1}\sigma_{r\theta} = \rho v_{tt}, \qquad (3b)$$

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$$\sigma_{rz,r} + r^{-1}\sigma_{\theta z,\theta} + \sigma_{zz,z} + r^{-1}\sigma_{r\theta} = \rho w_{,tt},$$
(3c)

$$\rho c_{v} \kappa \nabla^{2} T = \rho \tau T_{,tt} + \rho c_{v} T_{,t} - (Q + \tau Q_{,t}) + \beta T_{0} \frac{\partial}{\partial t} \Big[u_{,r} + r^{-1} (u + v_{,\theta}) + w_{,z} \Big],$$
(3d)

where ρ is the mass density, c_{ν} is the specific heat capacity, $\kappa = K / \rho c_{\nu}$ is the diffusity, K is the thermal conductivity, τ is the constant of generalized thermoelasticity, T_0 is the reference temperature. Substituting the Eqs. (1) and (2) in Eqs. (3), the following displacement equations of motion are obtained

$$(\lambda + 2\mu)(u_{,rr} + r^{-1}u_{,r} - r^{-2}u) + \mu r^{-2}u_{,\theta\theta} + \mu u_{,zz} + r^{-1}(\lambda + \mu)v_{,r\theta}$$

- $r^{-2}(\lambda + 3\mu)v_{,\theta} + (\lambda + \mu)w_{,rz} - \beta(T_{,r} + \eta T_{,rt}) = \rho u_{,tt},$ (4a)

$$\mu (v_{,rr} + r^{-1}v_{,r} - r^{-2}v) + r^{-2} (\lambda + 2\mu)v_{,\theta\theta} + \mu v_{,zz} + r^{-2} (\lambda + 3\mu)u_{,\theta} + r^{-1} (\lambda + \mu)u_{,r\theta} + r^{-1} (\lambda + \mu)w_{,\theta z} - \beta (T_{,\theta} + \eta T_{,\theta t}) = \rho v_{,tt},$$
(4b)

$$(\lambda + 2\mu)w_{,zz} + \mu(w_{,rr} + r^{-1}w_{,r} + r^{-2}w_{,\theta\theta}) + (\lambda + \mu)u_{,rz} + r^{-1}(\lambda + \mu)v_{,\theta z} + r^{-1}(\lambda + \mu)u_{,z} - \beta(T_{,z} + \eta T_{,zt}) = \rho w_{,tt}, \qquad (4c)$$

$$\rho c_{v} \kappa \Big(T_{,rr} + r^{-1} T_{,r} + r^{-2} T_{,\theta\theta} + T_{,zz} \Big) = \rho \tau T_{,tt} + \rho c_{v} T_{,t} - (Q + \tau Q_{,t}) + \beta T_{0} \Big[u_{,tr} + r^{-1} (u_{,t} + v_{,t\theta}) + w_{,tz} \Big].$$
(4d)

The heat conducting elastic rod is subjected to a moving heat source of constant strength releasing its energy continuously while moving along the positive direction of the z-axis with a constant velocity v. This moving heat source is assumed to be the following non-dimensional form

$$Q = Q_0 \delta(z - vt), \tag{5}$$

where Q_0 is a constant and δ is the delta function.

4. General solution technique

The Equations (4) are the coupled partial differential equations of the three displacements and heat conduction components. To uncouple the Eqs. (4), we follow Fourier series solutions discussed by Mirsky [2] as follows:

$$u(r,\theta,z,t) = \sum_{n=0}^{\infty} \varepsilon_n \left[\left(\phi_{n,r} + r^{-1} \psi_{n,\theta} \right) + \left(\overline{\phi}_{n,r} + r^{-1} \overline{\psi}_{n,\theta} \right) \right] e^{i(kz+\omega t)},$$
(6a)

$$v(r,\theta,z,t) = \sum_{n=0}^{\infty} \mathcal{E}_n \left[\left(r^{-1} \phi_{n,\theta} - \psi_{n,r} \right) + \left(r^{-1} \overline{\phi}_{n,\theta} - \overline{\psi}_{n,r} \right) \right] e^{i(kz+\omega t)},$$
(6b)

$$w(r,\theta,z,t) = (i/a) \sum_{n=0}^{\infty} \varepsilon_n \left[W_n + \overline{W}_n \right] e^{i(kz+\omega t)},$$
(6c)

$$T(r,\theta,z,t) = \left(\left(\lambda + 2\mu\right) / \beta a^2 \right) \sum_{n=0}^{\infty} \left[T_n + \overline{T}_n \right] e^{i(kz+\omega t)},$$
(6d)

where $\varepsilon_n = \frac{1}{2}$ for n = 0, $\varepsilon_n = 1$ for $n \ge 1$, $i = \sqrt{-1}$, k is the wave number, ω is the frequency, $\phi_n(r,\theta)$, $W_n(r,\theta)$, $T_n(r,\theta)$, $\psi_n(r,\theta)$, $\overline{\phi}_n(r,\theta)$, $\overline{W}_n(r,\theta)$, $\overline{T}_n(r,\theta)$, $\overline{\psi}_n(r,\theta)$ are the displacement potentials and a is the geometrical parameter of the cylinder.

Introducing the irrotational velocity $c_1^2 = (\lambda + 2\mu)/\rho$ and dimensionless quantities such

as
$$\zeta = ka$$
, $\overline{z} = z/a$, $T_a = t\sqrt{\mu/\rho}/a$, $x = r/a$, $\alpha' = c_1 a/\kappa$, $\Omega^2 = \omega^2 a^2/c_1^2$, $\chi_1 = \frac{T_0 a}{\rho^2 c_v c_1 \kappa} \beta^2$,

 $Q^* = \frac{Q_0 v}{i\chi_1 \Omega}, \ \chi_2 = \frac{c_1^2}{c_v \kappa} \tau, \ \chi_3 = \frac{c_1}{a} \eta, \text{ and } \ \chi_4 = 1/(2 + \overline{\lambda}), \text{ and using Eqs. (5) and (6) in Eqs. (4),}$ we obtain

$$\left(\nabla^2 + \Omega^2 - \chi_4 \varsigma\right) \phi_n - \varsigma \left(1 + \overline{\lambda}\right) \chi_4 W_n - \left(1 + i \chi_3 \Omega\right) T_n = 0, \qquad (7a)$$

$$\varsigma \left(1+\overline{\lambda}\right) \chi_4 \nabla^2 \phi_n + \left(\chi_4 \nabla^2 + \Omega^2 - \varsigma^2\right) W_n - \varsigma \left(1+i\chi_3 \Omega\right) T_n = 0, \qquad (7b)$$

$$-i\chi_{1}\Omega\nabla^{2}\phi_{n}+i\chi_{1}\varsigma\Omega W_{n}+\left(\nabla^{2}-i\alpha'\Omega+\chi_{2}-Q^{*}\right)T_{n}=0, \qquad (7c)$$

$$\left(\nabla^2 + \left(2 + \overline{\lambda}\right)\Omega^2 - \varsigma^2\right) \ \psi_n = 0, \qquad (7d)$$

where $\nabla^2 \equiv \partial^2 / \partial x^2 + x^{-1} \partial / \partial x + x^{-2} \partial^2 / \partial \theta^2$.

The parameters defined in Eqs. (7), namely, χ_1 couples the equations corresponding to the elastic wave propagation and the heat conduction which is called the coupling factor; the coefficient χ_2 , which is introduced by the theory of generalized thermoelasticity, may render the governing system of equations hyperbolic. The parameter χ_3 is the coefficient of the term indicating the difference between empirical and thermodynamic temperatures.

Rewriting Eqs. (7) results in the following vanishing determinant form

$$\begin{vmatrix} \nabla^{2} + \Omega^{2} - \chi_{4} \varsigma^{2} \end{pmatrix} - \zeta \chi_{4} \left(1 + \overline{\lambda} \right) & (1 + i \chi_{3} \Omega) \\ \zeta \chi_{4} \left(1 + \overline{\lambda} \right) & (\nabla^{2} \chi_{4} + \Omega^{2} - \varsigma^{2}) & -\zeta \left(1 + i \chi_{3} \Omega \right) \\ - i \chi_{1} \Omega & i \zeta \chi_{1} \Omega & (\nabla^{2} - i \alpha' \Omega + \chi_{2} - Q^{*}) \end{vmatrix} \begin{pmatrix} \phi_{n}, W_{n}, T_{n} \end{pmatrix} = 0.$$

$$(8)$$

Equation (8), on simplification reduces to the following differential equation

$$\left(A\nabla^{6} + B\nabla^{4} + C\nabla^{2} + D\right)\left(\phi_{n}, W_{n}, T_{n}\right) = 0, \qquad (9)$$

where

$$A = \chi_4, \tag{10a}$$

$$B = \chi_4 \left(\Omega^2 - \chi_4 \varsigma^2 + \chi_2 - i \alpha \Omega \right) + \varsigma^2 \chi_4^2 \left(1 + \overline{\lambda} \right)^2 + \left(\Omega^2 - \varsigma^2 \right),$$
(10b)
$$C = i \varsigma \Omega \left(1 + i \chi_3 \Omega \right) \left(\varsigma^2 + \varsigma^2 \chi_4 \left(1 + \overline{\lambda} \right) + \chi_4 \right)$$

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$$+ \left(\Omega^{2} - \chi_{4}\varsigma^{2}\right) \left(\Omega^{2} - \varsigma^{2} - \chi_{4}\left(\chi_{2} - i\Omega\alpha'\right)\right) \\ - \left(\chi_{2} - i\Omega\alpha'\right) \left(\left(\Omega^{2} - \varsigma^{2}\right) + \varsigma^{2}\chi_{4}^{2}\left(1 + \overline{\lambda}\right)\right) , \qquad (10c)$$

$$D = i\Omega\alpha' (1 + i\chi_3\Omega) \Big(-\varsigma^2 \chi_4 \Big(1 + \overline{\lambda} \Big) + \Big(\Omega^2 - \chi_4 \varsigma^2 \Big) \varsigma^2 + \Big(\Omega^2 - \varsigma^2 \Big) \Big) \\ + \Big(\chi_2 - i\Omega\alpha' \Big) \Big(\Omega^2 - \chi_4 \varsigma^2 \Big) \Big(\Omega^2 - \varsigma^2 \Big).$$
(10d)

Solving the Eq. (9), the solutions for the symmetric mode are obtained as

$$\phi_n = \sum_{j=1}^3 A_{jn} J_n \left(\alpha_j a x \right) \cos n\theta , \qquad (11a)$$

$$W_n = \sum_{j=1}^3 d_j A_{jn} J_n \left(\alpha_j a x \right) \cos n\theta , \qquad (11b)$$

$$T_n = \sum_{j=1}^3 e_j A_{jn} J_n \left(\alpha_j a x \right) \cos n\theta \,. \tag{11c}$$

Similarly, the solutions for the antisymmetric mode are obtained by changing $\cos n\theta$ by $\sin n\theta$ in Eqs. (11) and are given as

$$\overline{\phi}_n = \sum_{j=1}^3 \overline{A}_{jn} J_n(\alpha_j a x) \sin n\theta , \qquad (12a)$$

$$\overline{W}_n = \sum_{j=1}^3 d_j \overline{A}_{jn} J_n (\alpha_j a x) \sin n\theta , \qquad (12b)$$

$$\overline{T}_n = \sum_{j=1}^3 e_j \overline{A}_{jn} J_n(\alpha_j a x) \sin n\theta , \qquad (12c)$$

where $(\alpha_j a)^2 > 0$, (j = 1, 2, 3) are the roots of the following algebraic equation

$$A(\alpha a)^{6} - B(\alpha a)^{4} + C(\alpha a)^{2} + D = 0$$
⁽¹³⁾

and the constants d_j and e_j are given in Eqs. (12b) and (12c) can be calculated from the equations

$$\chi_4 \varsigma \left(1 + \overline{\lambda}\right) d_j + \left(1 + i\chi_3 \Omega\right) e_j = \left(\Omega^2 - \varsigma^2\right) \chi_4 - \left(\alpha_j a\right)^2, \tag{14a}$$

$$\left(\left(\Omega^{2}-\varsigma^{2}\right)-\left(\alpha_{j}a\right)^{2}\chi_{4}\right)d_{j}-\varsigma\left(1+i\chi_{3}\Omega\right)e_{j}=\left(\alpha_{j}a\right)^{2}\chi_{4}\varsigma\left(1+\overline{\lambda}\right).$$
(14b)

Solving the Eq. (7d), the solution to the symmetric mode is obtained as

$$\psi_n = A_{4n} J_n \left(\alpha_4 a x \right) \sin n\theta \,, \tag{15}$$

and for the antisymmetric mode is

$$\overline{\psi}_n = A_{4n} J_n(\alpha_4 a x) \cos n\theta , \qquad (16)$$

where $(\alpha_4 a)^2 = \Omega^2 - \varsigma^2$. If $(\alpha_j a)^2 < 0$, (j = 1, 2, 3), then the Bessel function J_n is to be replaced by the modified Bessel function I_n .

In this problem, the free vibration of a generalized thermoelastic solid bar of polygonal cross section is considered. Since the boundary is irregular, the Fourier expansion collocation method is applied on the boundary of the cross section. Thus, the boundary conditions obtained are

$$\left(\sigma_{pp}\right)_{l} = \left(\sigma_{pq}\right)_{l} = \left(\sigma_{zp}\right)_{l} = \left(T\right)_{l} = 0.$$
(17)

where p is the coordinate normal to the boundary and q is the coordinate in the tangential direction. Here σ_{pp} is the normal stress, σ_{pq} and σ_{zp} are the shearing stresses and ()_l is the value at the l-th segment of the boundary. Since the coordinate p and q are functions of r and θ , it is difficult to find transformed expressions for the stresses. Therefore the curved boundary is divided into small segments such that the variations of the stresses are assumed to be constant. Assuming the angle γ_l , between the normal to the segment and the reference axis to be constant, the transformed expressions for the stresses are followed by Nagaya [3-6]

$$\sigma_{pp} = 2\mu \Big[u_{,r} \cos^{2}(\theta - \gamma_{l}) + r^{-1}(u + v_{,\theta}) \sin^{2}(\theta - \gamma_{l}) + 0.5 \Big(r^{-1}[u - u_{,\theta}] - v_{,r} \Big) \sin 2(\theta - \gamma_{l}) \Big] \\ + \lambda \Big(u_{,r} + r^{-1}(u + v_{,\theta}) + w_{,z} \Big),$$
(18a)

$$\sigma_{pq} = \mu \left[(u_{,r} - r^{-1}(v_{,\theta} + u)) \sin 2(\theta - \gamma_l) + (r^{-1}(u_{,\theta} - v) + v_{,r}) \cos 2(\theta - \gamma_l) \right],$$
(18b)

$$\sigma_{zq} = \mu \left[(u_{,z} + w_{,r}) \cos(\theta - \gamma_l) - (v_{,z} + r^{-1} w_{,\theta}) \sin(\theta - \gamma_l) \right].$$
(18c)

Applying the Fourier expansion collocation method along the curved surface of the boundary, the transformed expressions for the stresses are

$$\left\lfloor \left(S_{pp}\right)_{l} + \left(\overline{S}_{pp}\right)_{l} \right\rfloor e^{i(\overline{\varsigma}z + \Omega T_{a})} = 0, \\ \left\lfloor \left(S_{pq}\right)_{l} + \left(\overline{S}_{pq}\right)_{l} \right\rfloor e^{i(\overline{\varsigma}z + \Omega T_{a})} = 0,$$
(19a,b)

$$\left\lfloor \left(S_{zp}\right)_{l} + \left(\overline{S}_{zp}\right)_{l} \right\rfloor e^{i(\varsigma \overline{z} + \Omega T_{a})} = 0, \\ \left\lfloor \left(S_{t}\right)_{l} + \left(\overline{S}_{t}\right)_{l} \right\rfloor e^{i(\varsigma \overline{z} + \Omega T_{a})} = 0,$$
(19c,d)

where,

$$S_{pp} = 0.5 \left(A_{10} e_0^1 + A_{20} e_0^2 + A_{30} e_0^3 + B_{50} e_0^5 \right) + \sum_{n=1}^{\infty} \left(A_{1n} e_n^1 + A_{2n} e_n^2 + A_{3n} e_n^3 + A_{4n} e_n^4 + B_{5n} e_n^5 \right),$$
(20a)

$$S_{pq} = 0.5 \left(A_{10} f_0^1 + A_{20} f_0^2 + A_{30} f_0^3 \right) + \sum_{n=1}^{\infty} \left(A_{1n} f_n^1 + A_{2n} f_n^2 + A_{3n} f_n^3 + A_{4n} f_n^4 \right),$$
(20b)

$$S_{zp} = 0.5 \left(A_{10} g_0^1 + A_{20} g_0^2 + A_{30} g_0^3 \right) + \sum_{n=1}^{\infty} \left(A_{1n} f_n^1 + A_{2n} f_n^2 + A_{3n} f_n^3 + A_{4n} f_n^4 \right),$$
(20c)

$$S_{t} = 0.5 \left(A_{10}k_{0}^{1} + A_{20}k_{0}^{2} + A_{30}k_{0}^{3} \right) + \sum_{n=1}^{\infty} \left(A_{1n}k_{n}^{1} + A_{2n}k_{n}^{2} + A_{3n}k_{n}^{3} \right),$$
(20d)

$$\overline{S}_{pp} = 0.5\overline{A}_{40}\overline{e}_{0}^{-4} + \sum_{n=1}^{\infty} \left(\overline{A}_{1n}\overline{e}_{n}^{-1} + \overline{A}_{2n}\overline{e}_{n}^{-2} + \overline{A}_{3n}\overline{e}_{n}^{-3} + \overline{A}_{4n}\overline{e}_{n}^{-4} + \overline{B}_{5n}\overline{e}_{n}^{-5} \right),$$
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$$\overline{S}_{pq} = 0.5\overline{A}_{40}\overline{f}_{0}^{4} + \sum_{n=1}^{\infty} \left(\overline{A}_{1n}\overline{f}_{n}^{1} + \overline{A}_{2n}\overline{f}_{n}^{2} + \overline{A}_{3n}\overline{f}_{n}^{3} + \overline{A}_{4n}\overline{f}_{n}^{4}\right),$$
(21b)

$$\overline{S}_{zp} = 0.5\overline{A}_{40}\overline{g}_{0}^{4} + \sum_{n=1}^{\infty} \left(\overline{A}_{1n}\overline{g}_{n}^{1} + \overline{A}_{2n}\overline{g}_{n}^{2} + \overline{A}_{3n}\overline{g}_{n}^{3} + \overline{A}_{4n}\overline{g}_{n}^{4} \right),$$
(21c)

$$\overline{S}_{t} = \sum_{n=1}^{\infty} \left(\overline{A}_{1n} \overline{k}_{n}^{1} + \overline{A}_{2n} \overline{k}_{n}^{2} + \overline{A}_{3n} \overline{k}_{n}^{3} \right).$$
(21d)

The functions $e_n^j \sim \overline{k}_n^j$ used in the boundary conditions of the symmetric and antisymmetric cases are given in Appendix A.



Fig. 1. Geometry of a straight line segment.

The boundary conditions along the entire range of the boundary cannot be satisfied directly. To satisfy the boundary conditions, the Fourier expansion collocation method due to Nagaya [3-6] is applied along the boundary. Performing the Fourier series expansion to the transformed expression in Eq. (14) along the boundary, the boundary conditions are expanded in the form of double Fourier series for symmetric and antisymmetric modes of vibrations. For the symmetric mode, the equation, which satisfies the boundary conditions, is obtained in matrix form as follows:

$$E_{mn}^{j} = \left(2\varepsilon_{n}/\pi\right)\sum_{l=1}^{L}\int_{\theta_{l-1}}^{\theta_{l}}e_{n}^{j}\left(R_{l},\theta\right)\cos m\theta d\theta, \qquad (24a)$$

$$F_{mn}^{j} = \left(2\varepsilon_{n}/\pi\right)\sum_{l=1}^{L}\int_{\theta_{l-1}}^{\theta_{l}}f_{n}^{j}\left(R_{l},\theta\right)\sin m\theta d\theta, \qquad (24b)$$

$$G_{mn}^{j} = \left(2\varepsilon_{n}/\pi\right)\sum_{l=1}^{L}\int_{\theta_{l-1}}^{\theta_{l}} g_{n}^{j}\left(R_{l},\theta\right)\cos m\theta d\theta, \qquad (24c)$$

$$K_{mn}^{j} = \left(2\varepsilon_{n}/\pi\right)\sum_{l=1}^{L}\int_{\theta_{l-1}}^{\theta_{l}}k_{n}^{j}\left(R_{l},\theta\right)\cos m\theta d\theta.$$
(24d)

Here j = 1, 2, 3, 4 and 5, L is the number of segments, R_i is the coordinate r at the boundary and N is the number of terms in the Fourier series.

The boundary conditions for the antisymmetric mode are written in the form of a matrix as given below:

where,

$$\overline{E}_{nm}^{j} = \left(2\varepsilon_{n}/\pi\right)\sum_{l=1}^{L}\int_{\theta_{l-1}}^{\theta_{l}}\overline{e}_{n}^{j}\left(R_{l},\theta\right)\sin m\theta d\theta, \ \overline{F}_{mn}^{j} = \left(2\varepsilon_{n}/\pi\right)\sum_{l=1}^{L}\int_{\theta_{l-1}}^{\theta_{l}}\overline{f}_{n}^{j}\left(R_{l},\theta\right)\cos m\theta d\theta, \ (26a,b)$$

$$\overline{G}_{mn}^{j} = \left(2\varepsilon_{n}/\pi\right)\sum_{l=1}^{L}\int_{\theta_{l-1}}^{\theta_{l}}\overline{g}_{n}^{j}\left(R_{l},\theta\right)\sin m\theta d\theta, \ \overline{K}_{mn}^{j} = \left(2\varepsilon_{n}/\pi\right)\sum_{l=1}^{L}\int_{\theta_{l-1}}^{\theta_{l}}\overline{k}_{n}^{j}\left(R_{l},\theta\right)\sin m\theta d\theta.$$
(26c,d)

5. Numerical results and discussion

In accordance with the theoretical results obtained in the previous sections and comparing these results with the literature results, some numerical analysis of the dispersion equation is carried out for triangular, square, pentagonal and hexagonal cross sectional bars. The secant method given by Antia (2002) is used to obtain the roots of the frequency equation. The material properties of copper at temperature 4.2 *K* are taken approximately as Poisson ratio v = 0.3, the Young's modulus $E = 2.139 \times 10^{11} \text{ N/m}^2$, $\lambda = 8.20 \times 10^{11} \text{ kg/ms}^2$, $\mu = 4.20 \times 10^{10} \text{ kg/ms}^2$, $c_v = 9.1 \times 10^{-2} \text{ m}^2/\text{ks}^2$, $K = 113 \times 10^2 \text{ kgm/ks}^2$, $\rho = 8.96 \times 10^3 \text{ kg/m}^3$

and $Q_0 = 10$. The other parameters such as α' , χ_1 , χ_2 , and χ_3 are chosen by following the arguments given by Erbay and Suhubi [14].

The geometric relations for the polygonal cross-sections given as

$$R_i/b = \left[\cos\left(\theta - \gamma_i\right)\right]^{-1},\tag{27}$$

where b is the apothem. The relation given in Eq. (23) is used directly for the numerical calculation. The dimensionless wave numbers, which are complex in nature, are computed by fixing Ω for $0 < \Omega \le 1$ using secant method (applicable for complex roots [20]). The basic independent modes like longitudinal and flexural vibrations of bar are analyzed and the corresponding non-dimensional wave numbers are computed. The polygonal cross-sectional bar in the range $\theta = 0$ and $\theta = \pi$ is divided into many segments for convergence of wave number in such a way that the distance between any two segments is negligible. The computation of Fourier coefficients given in Eq. (21) is carried out using the five-point Gaussian quadrature method. To obtain the roots of the frequency equation the secant method applicable for the complex roots is employed. The results of longitudinal and flexural (antisymmetric) modes are plotted in the form of dispersion curves. The notations used in the figures namely, LM denotes the longitudinal mode and 2 refer the second mode in all the dispersion curves.

In Tables 1-2, a comparison is made between the real $R(\xi)$ and imaginary $I(\xi)$ parts of frequency for longitudinal modes of the polygonal cross-sectional bar with different moving heat source velocities v=1, v=2 and v=3. From these tables it is clear that as the vibration modes increases, the non dimensional frequencies are also increases in all the four cross sectional bars and also it is clear that the non dimensional frequency profiles exhibits low energy for increasing heat source velocity.

Table 1. Comparison between the real $R(\xi)$ and imaginary $I(\xi)$ parts of frequency for longitudinal modes of the Triangular and Pentagonal cross-sectional bar with moving heat source velocities v=1, v=2 and v=3.

	Triangle							Pentagon						
Mode	v = 1		<i>v</i> =	= 2	<i>v</i> =	= 3	<i>v</i> =	=1	v = 2		<i>v</i> = 3			
	$R(\xi)$	$I(\xi)$	$R(\xi)$	$I(\xi)$	$R(\xi)$	$I(\xi)$	$R(\xi)$	$I(\xi)$	$R(\xi)$	$I(\xi)$	$R(\xi)$	$I(\xi)$		
1	0.0851	0.0844	0.0884	0.0879	0.0817	0.0813	0.0876	0.0849	0.0896	0.0880	0.0826	0.0830		
2	0.2627	0.2603	0.2611	0.2634	0.2470	0.2497	0.2744	0.2707	0.2782	0.2671	0.2484	0.2533		
3	0.4615	0.4508	0.4502	0.4449	0.4231	0.4249	0.4624	0.4510	0.4668	0.4549	0.4217	0.4287		
4	0.6562	0.6366	0.6553	0.6406	0.6089	0.5983	0.6505	0.6246	0.6560	0.6391	0.6030	0.5997		
5	0.9548	0.9179	0.9508	0.9221	0.8974	0.8636	0.9606	0.9096	0.9477	0.9090	0.8832			

Another spectrum of the non-dimensional frequencies of flexural (antisymmetric) modes of four cross sectional bars is discussed in Tables 3-4. From Tables 3-4, it is observed that as the modes are increases the non-dimensional frequency also increases, whereas the dispersion of flexural modes are almost exponentially decreasing with increasing values of heat source velocity. The amplitude of the all modes of vibrations possess high energy in flexural (antisymmetric) compared with longitudinal modes due to the temperature changes are more pertinent in flexural (antisymmetric) modes.

Table 2. Comparison between the real $R(\xi)$ and imaginary $I(\xi)$ parts of frequency for longitudinal modes of the Square and Hexagonal cross-sectional bar with moving heat source velocities v=1, v=2 and v=3.

	Square							Hexagon						
Mode	v = 1		<i>v</i> =	= 2	<i>v</i> =	= 3	<i>v</i> =	=1	$1 \qquad v = 2$		<i>v</i> = 3			
	$R(\xi)$	$I(\xi)$	$R(\xi)$	$I(\xi)$	$R(\xi)$	$I(\xi)$	$R(\xi)$	$I(\xi)$	$R(\xi)$	$I(\xi)$	$R(\xi)$	$I(\xi)$		
1	0.0914	0.0914	0.0904	0.0903	0.0842	0.0880	0.0942	0.0825	0.1002	0.0998	0.0957	0.1030		
2	0.2717	0.2705	0.2732	0.2697	0.2619	0.2621	0.2766	0.2811	0.2984	0.3017	0.3717	0.2856		
3	0.4594	0.4551	0.4611	0.4491	0.4397	0.4351	0.4622	0.4545	0.4993	0.5002	0.5014	0.5016		
4	0.6466	0.6349	0.6526	0.6308	0.6209	0.6093	0.6530	0.6361	0.7081	0.7010	0.7084	0.7223		
5	0.9564	0.9315	0.9429	0.8937	0.8971	0.8758	0.9168	0.8881	1.0098	0.9804	1.0076			

Table 3. Comparison between the real $R(\xi)$ and imaginary $I(\xi)$ parts of frequency for flexural antisymmetric modes of the Triangle and Pentagonal cross-sectional bar with moving heat source velocities v = 1, v = 2 and v = 3.

	Triangle							Pentagon						
Mode	v = 1		<i>v</i> =	= 2	<i>v</i> =	= 3	<i>v</i> =	= 1	<i>v</i> = 2		<i>v</i> = 3			
	$R(\xi)$	$I(\xi)$	$R(\xi)$	$I(\xi)$	$R(\xi)$	$I(\xi)$	$R(\xi)$	$I(\xi)$	$R(\xi)$	$I(\xi)$	$R(\xi)$	$I(\xi)$		
1	0.0974	0.0870	0.0892	0.0889	0.0748	0.0753	0.0891	0.0889	0.0891	0.0889	0.0625	0.0646		
2	0.2665	0.1596	0.2726	0.2677	0.2242	0.2353	0.2690	0.2655	0.2651	0.2591	0.1820	0.2254		
3	0.4529	0.4357	0.4592	0.4488	0.4916	0.4023	0.4545	0.4401	0.4527	0.4296	0.3467	0.4148		
4	0.7434	0.6147	0.6486	0.6316	0.5722	0.5590	0.6477	0.6170	0.6512	0.6220	0.5294	0.5783		
5	0.9430	0.9012	0.9995	0.9409	0.8600	0.7878	0.9432	0.8914	0.9466	0.9003	0.7998			

Table 4. Comparison between the real $R(\xi)$ and imaginary $I(\xi)$ parts of frequency for flexural antisymmetric modes of the Square and Hexagonal cross-sectional bar moving heat source velocities v=1, v=2 and v=3.

	Square							Hexagon						
Mode	v = 1		<i>v</i> =	= 2	<i>v</i> =	= 3	<i>v</i> =	=1	v = 2		<i>v</i> = 3			
	$R(\xi)$	$I(\xi)$	$R(\xi)$	$I(\xi)$	$R(\xi)$	$I(\xi)$	$R(\xi)$	$I(\xi)$	$R(\xi)$	$I(\xi)$	$R(\xi)$	$I(\xi)$		
1	0.1001	0.1001	0.0561	0.0345	0.0999	0.0999	0.1000	0.1000	0.0976	0.0973	0.0999	0.1001		
2	0.3000	0.3000	0.3005	0.3004	0.3002	0.3002	0.3000	0.3000	0.5000	0.5000	0.3002	0.3006		
3	0.5004	0.5004	0.5003	0.5002	0.5005	0.5005	0.4785	0.4785	0.6200	0.6032	0.5004	0.5005		
4	0.7007	0.7007	0.7008	0.6993	0.7059	0.7004	0.7007	0.7007	0.9870	0.8765	0.7007	0.7007		
5	1.0019	1.0008	1.0010	1.0010	0.99999	1.0004	1.0000	1.0000	1.0010	1.0010	1.0201			

Comparing the results of uniform cross sectional bars subjected to moving heat source, it is noticed that as the different vibrational modes increases, the real part of the nondimensional frequency $R(\xi)$ obtained for the polygonal cross sectional bar increases whereas the imaginary part $I(\xi)$ decreases, which is the proper physical behavior that the dissipation of energy due to moving heat sources.

5.1. Triangular and pentagonal cross-sections. In the triangular and pentagonal cross-sectional cross sectional bar, the vibrational displacements are symmetrical about the x axis for the longitudinal mode and antisymmetrical about the y axis for the flexural mode since the cross-section is symmetric about only one axis. Therefore n and m are chosen as 0, 1, 2, 3... in Eq. (23) for the longitudinal mode and n, m=1, 2, 3 ... in Eq. (25) for the flexural mode and the dimensionless frequency Ω are calculated by fixing complex wave number $|\varsigma|$. The variation of non-dimensional frequency with dimensionless wave number of a triangular cross sectional bar are shown in Fig. 2 and Fig. 3 with respect to the heat source distance z = 1, z = 2 and constant heat source velocity v. From the Fig. 2, it is observed that, the propagation of energy is linear for the longitudinal and flexural (anti symmetric) modes of the bar.

The wave propagation of bar with increasing heat source distance in Fig. 3, the behavior is though linear, becomes saturated quickly for the longitudinal mode beyond $|\varsigma| = 0.2$ and for the flexural (antisymmetric) mode beyond $|\varsigma| = 0.4$.



Fig. 2. Non-dimensional frequency versus dimensionless wave number of a triangular cross sectional bar for z = 1 with $\alpha' = 2.0$, $\chi_1 = 2.6 \times 10^{-7}$, $\chi_2 = \chi_3 = 1.0$.



Fig. 3. Non dimensional frequency versus dimensionless wave number of a triangular cross sectional bar for z = 2 with $\alpha' = 2.0$, $\chi_1 = 2.6 \times 10^{-7}$, $\chi_2 = \chi_3 = 1.0$.

In Figures 4 and 5, the variation of non-dimensional frequency with dimensionless wave number of a pentagonal cross sectional bar is discussed with respect to the heat source distance z = 1 and z = 2. Here, it is observed that, the dispersion is slightly high in both

modes of bar with respect to the heat source distance z=1 and z=2 than in the case of triangular cross sectional bar. The frequencies increase for higher modes of vibrations, and the cross over points in the trend line indicates the transfer of heat energy between the modes of vibrations.



Fig. 4. Non dimensional frequency versus dimensionless wave number of a pentagonal bar for z = 1 with $\alpha' = 2.0$, $\chi_1 = 2.6 \times 10^{-7}$, $\chi_2 = \chi_3 = 1.0$.



Fig. 5. Non dimensional frequency versus dimensionless wave number of a pentagonal bar for z = 2 with $\alpha' = 2.0$, $\chi_1 = 2.6 \times 10^{-7}$, $\chi_2 = \chi_3 = 1.0$.

5.2. Square and Hexagonal cross-sections. In case of longitudinal vibration of square and hexagonal cross-sectional bars, the displacements are symmetrical about both major and minor axes, since both the cross-sections are symmetric about both the axes. Therefore the frequency equation is obtained by choosing both terms of n and m as 0, 2, 4, 6... in Eq. (23). During flexural motion, the displacements are antisymmetrical about the major axis and symmetrical about the minor axis. Hence the frequency equation is obtained by choosing n, m=1, 3, 5 in Eq. (25).

A graph is drawn between the non-dimensional frequency Ω versus dimensionless wave number $|\varsigma|$ of transversely isotropic square cross sectional bar for longitudinal and flexural (antisymmetric) modes of vibrations with the heat source distance z=1 and z=2 which is shown in Figs. 6 and 7. From Figures 6 and 7, it is clear that, the displacement of energy in the first mode and second mode of vibrations of longitudinal and flexural (antisymmetric) increases linearly as the wave number increases. It is also observed that at $|\zeta| = 0.2$, the longitudinal and flexural (antisymmetric) modes of vibration merges, beyond $|\zeta| = 0.3$, the flexural and anti symmetric modes of vibrations are getting dispersed.

Figures 8 and 9 show that the non-dimensional frequency Ω versus dimensionless wave number $|\zeta|$ of transversely isotropic hexagonal cross sectional bar for longitudinal and flexural (antisymmetric) modes of vibrations with the heat source distance z = 1 and z = 2. It is observed that as the wave number increases, the non-dimensional frequency also increases linearly but, beyond $|\zeta|=0.2$, the first two modes of longitudinal and flexural (antisymmetric) have oscillating behavior. The transfer of heat energy is higher in the lower modes of vibrations as compared to the higher modes. This cross over point represents the transfer of heat energy between modes of vibration of square and hexagon.



Fig. 6. Non dimensional frequency versus dimensionless wave number of a square bar for z = 1 with $\alpha' = 2.0$, $\chi_1 = 2.6 \times 10^{-7}$, $\chi_2 = \chi_3 = 1.0$.



Fig. 7. Non dimensional frequency versus dimensionless wave number of a square bar for z = 2 with $\alpha' = 2.0$, $\chi_1 = 2.6 \times 10^{-7}$, $\chi_2 = \chi_3 = 1.0$.

We have shown that the frequencies depend strongly on the cross-sections of the bar and deviate from the circular one. The dispersion curves of higher wave numbers are sensitive to the nature of the thermal boundary condition as well as to the measure of the thermomechanical cross-coupling for both propagating and evanescent waves. These dispersions curves obtained from the exact solution of the problem could be used as references data for developing of reliable finite elements and boundary elements for approximate solution of the problems of wave propagation in structures with moving heat sources.



Fig. 8. Non dimensional frequency versus dimensionless wave number of a hexagonal bar for z = 1 with $\alpha' = 2.0, \chi_1 = 2.6 \times 10^{-7}, \chi_2 = \chi_3 = 1.0$.



Fig. 9. Non dimensional frequency versus dimensionless wave number of a hexagonal bar for z = 2 with $\alpha' = 2.0$, $\chi_1 = 2.6 \times 10^{-7}$, $\chi_2 = \chi_3 = 1.0$.

6. Conclusions

In this paper, the dynamic response of a heat conducting solid bar of polygonal cross section subjected to moving heat source is analyzed by satisfying the boundary conditions on the irregular boundary using the Fourier expansion collocation method and the frequency equation for the longitudinal and flexural vibrations are obtained. Numerically the frequency equations are analyzed for the bar of different cross-section such as triangular, square, pentagonal and hexagonal. From the results of the present method, it is clear that moving heat source and the different geometrical cross sections of the bar influence the frequency. The problem can be analyzed for any other cross section by using the proper geometric relation.

Appendix A

The expressions $e_n^j \sim \overline{k}_n^j$ used in Eqs. (19) and (21) are given as follows:

$$e_n^j = 2\varepsilon_4 \left\{ n(n-1)J_n(\alpha_j ax) + (\alpha_j ax)J_{n+1}(\alpha_j ax) \right\} \cos 2(\theta - \gamma_1) \cos n\theta$$

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$$- x^{2} \left\{ \varepsilon_{4}(\alpha_{j}a)^{2} + \left[\overline{\lambda} + 2\cos^{2}(\theta - \gamma_{l})\right] + \overline{\lambda}d_{j}\varepsilon_{4}\varsigma \right\} J_{n}(\alpha_{j}ax) \cos n\theta \\ + 2n\varepsilon_{4} \left\{ (n-1)J_{n}(\alpha_{j}ax) - (\alpha_{j}ax)J_{n+1}(\alpha_{j}ax) \right\} \sin n\theta \sin 2(\theta - \gamma_{l}), \quad j = 1, 2, 3$$
(A.1)

$$e_n^4 = 2\varepsilon_4 \{n(n-1)J_n(\alpha_4 ax) - (\alpha_4 ax)J_{n+1}(\alpha_4 ax)\} \cos n\theta \quad \cos 2(\theta - \gamma_1)$$

+2\varepsilon_4 \left[n(n-1) - (\alpha_4 ax)^2]J_n(\alpha_4 ax) + (\alpha_4 ax)J_{n+1}(\alpha_4 ax)\right] \sin n\theta \sin 2(\theta - \gamma_1), (A.2)

$$f_{n}^{j} = 2 \Big\{ [n(n-1) - (\alpha_{j}ax)^{2}] J_{n}(\alpha_{j}ax) + (\alpha_{j}ax) J_{n+1}(\alpha_{j}ax) \Big\} \cos n\theta \sin 2(\theta - \gamma_{l}) \\ + 2n \Big\{ (\alpha_{j}ax) J_{n+1}(\alpha_{j}ax) - (n-1) J_{n}(\alpha_{j}ax) \Big\} \sin n\theta \cos 2(\theta - \gamma_{l}), \ j = 1, 2, 3$$
(A.3)

$$f_n^4 = 2n\varepsilon_4 \left\{ (n-1)J_n(\alpha_4 ax) - (\alpha_4 ax)J_{n+1}(\alpha_4 ax) \right\} \cos n\theta \sin 2(\theta - \gamma_1)$$
$$-\varepsilon_4 \left\{ 2(\alpha_4 ax)J_{n+1}(\alpha_4 ax) - [(\alpha_4 ax)^2 - 2n(n-1)]J_n(\alpha_4 ax) \right\} \sin n\theta \cos 2(\theta - \gamma_1), \qquad (A.4)$$

$$g_n^{j} = (\varsigma + d_j) \Big\{ n J_n(\alpha_j a x) \cos(\overline{n - 1}\theta + \gamma_l) - (\alpha_j a x) J_{n+1}(\alpha_j a x) \cos(\theta - \gamma_l) \cos n\theta \Big\}, \quad j = 1, 2, 3$$
(A.5)

$$g_n^4 = \varsigma \left\{ nJ_n(\alpha_4 ax)\cos(\overline{n-1}\theta + \gamma_l) - (\alpha_4 ax)J_{n+1}(\alpha_4 ax)\sin n\theta\sin(\theta - \gamma_l) \right\}$$
(A.6)

$$k_n^j = e_j \left\{ n \cos\left(\overline{n-1}\theta + \gamma_l\right) J_n\left(\alpha_j a x\right) - \left(\alpha_j a x\right) J_{n+1}\left(\alpha_j a x\right) \cos\left(\theta - \gamma_l\right) \cos n\theta \right\}, \ j = 1, 2, 3$$
(A.7)

$$\overline{e}_{n}^{j} = 2\varepsilon_{4} \left\{ n(n-1)J_{n}(\alpha_{j}ax) + (\alpha_{j}ax)J_{n+1}(\alpha_{j}ax) \right\} \cos 2(\theta - \gamma_{l}) \sin n\theta$$

$$- x^{2} \left\{ \varepsilon_{4}(\alpha_{j}a)^{2} + \left[\overline{\lambda} + 2\cos^{2}(\theta - \gamma_{l})\right] + \overline{\lambda}d_{j}\varepsilon_{4}\varsigma \right\} J_{n}(\alpha_{j}ax) \cos n\theta$$

$$-2n\varepsilon_{4} \left\{ (n-1)J_{n}(\alpha_{j}ax) - (\alpha_{j}ax)J_{n+1}(\alpha_{j}ax) \right\} \cos n\theta \sin 2(\theta - \gamma_{l}), \ j = 1, 2, 3$$
(A.8)

$$\frac{e_{n}^{4}}{e_{n}} = 2\varepsilon_{4} \{n(n-1)J_{n}(\alpha_{4}ax) - (\alpha_{4}ax)J_{n+1}(\alpha_{4}ax)\} \sin n\theta \quad \cos 2(\theta - \gamma_{l})
-2\varepsilon_{4} \{[n(n-1) - (\alpha_{4}ax)^{2}]J_{n}(\alpha_{4}ax) + (\alpha_{4}ax)J_{n+1}(\alpha_{4}ax)\} \cos n\theta \sin 2(\theta - \gamma_{l}), \quad (A.9)$$

$$\overline{f}_{n}^{j} = 2\left\{ [n(n-1) - (\alpha_{j}ax)^{2}]J_{n}(\alpha_{j}ax) + (\alpha_{j}ax)J_{n+1}(\alpha_{j}ax)\right\} \sin n\theta \sin 2(\theta - \gamma_{l}) - 2n\left\{ (\alpha_{j}ax)J_{n+1}(\alpha_{j}ax) - (n-1)J_{n}(\alpha_{j}ax)\right\} \cos n\theta \cos 2(\theta - \gamma_{l}), \ j = 1, 2, 3$$
(A.10)

$$\overline{f}_{n}^{4} = 2n\varepsilon_{4}\left\{ (n-1)J_{n}(\alpha_{4}ax) - (\alpha_{4}ax)J_{n+1}(\alpha_{4}ax)\right\} \sin n\theta \sin 2(\theta - \gamma_{l})$$

+ $\varepsilon_{4}\left\{ 2(\alpha_{4}ax)J_{n+1}(\alpha_{4}ax) - [(\alpha_{4}ax)^{2} - 2n(n-1)]J_{n}(\alpha_{4}ax)\right\} \cos n\theta \cos 2(\theta - \gamma_{l}),$ (A.11)

$$\overline{g}_{n}^{j} = (\varsigma + d_{j}) \left\{ n J_{n}(\alpha_{j} a x) \cos(\overline{n - l}\theta + \gamma_{l}) - (\alpha_{j} a x) J_{n+1}(\alpha_{j} a x) \cos(\theta - \gamma_{l}) \sin n\theta \right\}, j = 1, 2, 3$$
(A.12)

$$\overline{g}_{n}^{-4} = \varsigma \left\{ n J_{n}(\alpha_{4} a x) \cos(\overline{n-1}\theta + \gamma_{l}) + (\alpha_{4} a x) J_{n+1}(\alpha_{4} a x) \cos n\theta \sin(\theta - \gamma_{l}) \right\},$$
(A.13)

$$\bar{k}_{n}^{j} = e_{j} \left\{ n \cos\left(\overline{n-1}\theta + \gamma_{l}\right) J_{n}\left(\alpha_{j}ax\right) + \left(\alpha_{j}ax\right) J_{n+1}\left(\alpha_{j}ax\right) \cos\left(\theta - \gamma_{l}\right) \sin n\theta \right\}, \quad j = 1, 2, 3.$$
(A.14)

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