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THEORY OF HYPERBOLIC TWO-TEMPERATURE GENERALIZED THERMOELASTICITY

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Abstract. Youssef improved the generalized thermoelasticity base on two distinct temperatures; the conductive temperature and the thermodynamics temperature which coincide together when the heat supply vanishes [1, 2]. This theory has one paradox, where it offers an infinite speed of thermal wave propagation. So, this work assuming a new consideration of the two types of temperature which depends upon the acceleration of the conductive and the thermal temperature. This work introduces the proof of the uniqueness of the solution, moreover, one dimensional numerical application. According to the numerical result this new model of thermoelasticity offers finite speed of thermal wave and mechanical wave propagation.

Keywords: elasticity, thermoelasticity, hyperbolic two-temperature, finite speed, wave propagation

1. Introduction

Duhamel was the first to consider elastic problems with heat changes. Neumann re-derived the equations obtained by Duhamel. This theory of uncoupled thermoelasticity consists of the heat equation independent of mechanical effects, and the equation of motion contains the temperature, as a known function. Danilovskaya [3] was the first who solved a problem in the context of the theory of uncoupled thermoelasticity with uniform heat, and it was for a half-space subjected to a thermal shock. There are two defects of this theory. This theory states that the mechanical state of the elastic body does not affect the temperature, which is not in accord with right physical experiments. Second, the heat equation being parabolic predicts an infinite speed of propagation for the temperature, which again contradicts physical observations.

Biot [4] introduced the coupled theory of thermoelasticity in which the equations of elasticity and heat conduction became coupled, and that agree with physical experiments, and any change of the temperature gives a certain amount of deformation in an elastic body and vice versa. The theory of coupled thermoelasticity has proved useful for many problems. The governing equations of this theory contain the equation of motion, which is a hyperbolic partial differential equation, and of the equation of energy conservation, which is parabolic. The nature of the heat equation implies that if an elastic medium is extending to infinity subjected to a thermal or mechanical disturbance, the effect will fall instantaneously at infinity, which contradicts physical experiments. Hence, a new equation of energy with hyperbolic type is needed.

Lord and Shulman [5] introduced the theory of generalized thermoelasticity with one

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relaxation time for the particular case of an isotropic body. Dhaliwal and Sherief [6] extended this theory to include the anisotropic case. In this theory, a modified law of heat conduction including both the heat flux and its time derivative replaces the conventional Fourier's law (Cattaneo's heat conduction). The heat equation associated is hyperbolic and hence eliminates the paradox of infinite speeds of propagation inherent in both

The second generalization of the coupled theory of elasticity is the theory of thermoelasticity with two relaxation times. Müller [7] in a review of the thermodynamics of thermoelastic solids, suggested an entropy production inequality, with the use of which he considered restrictions on a class of constitutive equations. Green and Laws [8] proposed a generalization of this inequality. Green and Lindsay [9] got an explicit version of the constitutive equations. These equations were obtained independently by Suhubi [10]. This theory contains two parameters that act as relaxation times. The classical Fourier's law of heat conduction is not satisfied if the medium under consideration has a center of symmetry.

Chen and Gurtin [11], Chen et al. [12, 13] have constructed a theory of heat conduction in deformable bodies, which depends upon two different temperatures, the conductive temperature, and the thermodynamic temperature. For time-independent situations, the difference between these two temperatures is proportional to the heat supply. In the absence of the heat supply, the two temperatures are identical. For time-dependent problems, however, and for wave propagation problems in particular, the two temperatures are in general different regardless of the presence of heat supply. The thermodynamic temperature, conductive temperature, and the strain are found to have representations in the form of a traveling wave plus a response, which happen instantaneously throughout the body [14]. Warren and Chen [15] investigated the wave propagation in the two-temperature theory of thermoelasticity.

Youssef [1] introduced a new theory of two-temperature generalized thermoelasticity with the general uniqueness theorem for the boundary mixed initial value problems in this theory. Youssef constructed a new theory of two-temperature generalized thermoelasticity theory for the homogeneous and isotropic body without energy dissipation; he presented the general uniqueness theory for the initial mixed boundary value problems in this theory [2], and he derived its variational principle [16].

2. Basic Equations

The governing equations of an isotropic and homogeneous thermoelastic medium, as proposed by Lord and Shulman are [5]:

The equation of motion

$$\sigma_{ij,j} + F_i = \rho\, \ddot{u}_i , \tag{1}$$

where σ_{ij} is the stress tensor, F_i is the body force components, ρ is the density, and u_i is the displacement components.

The constitutive relation is:

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e_{kk}\delta_{ij} - \gamma\left(T - T_0\right)\delta_{ij}, \tag{2}$$

where $\gamma = \left(3\lambda + 2\mu\right)\alpha_T$ are the coupling parameters, T is the dynamical temperature and T_0 being the reference temperature, e_{ij} is the strain tensor and λ and μ are the elastic constants of the material.

The deformation

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right). \tag{3}$$

The non-Fourier heat conduction

$$q_i + \tau_o \frac{\partial q_i}{\partial t} = -K\phi_{,i}, \tag{4}$$

where ϕ is the conductive temperature.

Moreover, we have

$$\rho C_E \dot{T} + \gamma T_0 e_{kk} = -q_{,i}, \tag{5}$$

where K is the thermal conductivity, q_i is the heat flux components, and C_E is the specific heat with constant strain .

The increment of the entropy η satisfies the following equations:

$$q_{i,i} = -\rho T_0 \dot{\eta}, \tag{6}$$

and

$$\rho T_0 \eta = \rho C_E T + T_0 \gamma_{ij} e_{ij}. \tag{7}$$

Equations (4), (6) and (7) formulate the heat conduction equations as proposed by Youssef [1] in the form:

$$K\phi_{,ii} = \left(\frac{\partial}{\partial t} + \tau_o \frac{\partial^2}{\partial t^2}\right)\left(\rho C_E T + \gamma T_0 e_{kk}\right), \tag{8}$$

and

$$\phi - T = a\phi_{,ii}, \tag{9}$$

where $a \geq 0$ is called the two-temperature parameter, while $i, j, k = 1, 2, 3$ are the indeces for any general co-ordinates in 3-dimensons.

3. One-Dimensional Generalized Thermoelastic Half-Space (Classical Two-Temperature)

Without losing the generality, we will consider one-dimensional isotropic and homogeneous thermoelastic medium occupies the half-space $x \geq 0$, and this medium is at rest in the undeformed state at zero time with uniform temperature T_0 .

When $t > 0$, the boundary $x = 0$ of the half-space subjected to a uniformly distributed time-dependent strain and temperature, then, the governing equations take the following forms:

$$(\lambda + 2\mu)\frac{\partial^2 e}{\partial x^2} - \gamma \frac{\partial^2 T}{\partial x^2} = \rho \frac{\partial^2 e}{\partial t^2}, \tag{10}$$

$$K\frac{\partial^2 \phi}{\partial x^2} = \left(\frac{\partial}{\partial t} + \tau_o \frac{\partial^2}{\partial t^2}\right)\left(\rho C_E T + \gamma T_0 e\right), \tag{11}$$

$$\phi - T = a\frac{\partial^2 \phi}{\partial x^2}, \tag{12}$$

$$\sigma = (\lambda + 2\mu)e - \gamma\left(T - T_0\right), \tag{13}$$

and

$$e = \frac{\partial u}{\partial x}. \tag{14}$$

The boundary and the initial conditions are

$$T(x,0) = \phi(x,0) = e(x,0) = \dot{T}(x,0) = \dot{\phi}(x,0) = \dot{e}(x,0) = 0, \tag{15}$$

and

$$e(0,t) = e_o(t), \quad \phi(0,t) = \phi_o(t). \tag{16}$$

For simplicity, use the following non-dimensional variables;

$$\left(x',u'\right)=c_o\eta\left(x,u\right),\ \left(t',\tau_o'\right)=c_o^2\eta\left(t,\tau_o\right),\ \theta=\frac{T-T_0}{T_0},\ \phi'=\frac{\phi-T_0}{T_0},\ \sigma'=\frac{\sigma}{\left(\lambda+2\mu\right)},\ c_o^2=\frac{\lambda+2\mu}{\rho},\ \eta=\frac{\rho C_E}{K}. \tag{17}$$

Hence,

$$\frac{\partial^2 e}{\partial x^2}-\varepsilon_1\frac{\partial^2\theta}{\partial x^2}=\frac{\partial^2 e}{\partial t^2}, \tag{18}$$

$$\frac{\partial^2\phi}{\partial x^2}=\left(\frac{\partial}{\partial t}+\tau_o\frac{\partial^2}{\partial t^2}\right)\left(\theta+\varepsilon_2 e\right), \tag{19}$$

$$\phi-\theta=\beta\frac{\partial^2\phi}{\partial x^2}, \tag{20}$$

and

$$\sigma=e-\varepsilon_1\theta, \tag{21}$$

where $\varepsilon_1=\dfrac{\gamma T_0}{\lambda+2\mu},\varepsilon_2=\dfrac{\gamma}{c_o^2\eta},\ \beta=c_o^2\eta^2 a,\ \varepsilon_1\ge 0,\ \varepsilon_2\ge 0,\ \beta\ge 0$.

Taking the Laplace transform for the both sides of the equations (18)-(21) as follows:

$$\bar f(s)=\int_0^\infty f\left(t\right)e^{-st}\,dt. \tag{22}$$

Hence,

$$\frac{d^2\bar e}{dx^2}-\varepsilon_1\frac{d^2\bar\theta}{dx^2}=s^2\bar e, \tag{23}$$

$$\frac{d^2\bar\phi}{dx^2}=\left(s+\tau_o s^2\right)\left(\bar\theta+\varepsilon_2\bar e\right), \tag{24}$$

$$\bar\phi-\bar\theta=\beta\frac{d^2\bar\phi}{dx^2}, \tag{25}$$

$$\bar\sigma=\bar e-\varepsilon_1\bar\theta, \tag{26}$$

$$\bar e=\frac{d\bar u}{dx}, \tag{27}$$

and

$$\bar e\left(0,s\right)=\bar e_o\left(s\right),\quad \bar\phi\left(0,s\right)=\bar\phi_o\left(s\right). \tag{28}$$

Eliminating $\bar\theta$ from equations (23)-(25), then

$$\frac{d^2\bar e}{dx^2}=\alpha_2\bar\phi+\alpha_3\bar e, \tag{29}$$

and

$$\frac{d^2\bar\phi}{dx^2}=\alpha_1\bar\phi+\varepsilon_2\alpha_1\bar e, \tag{30}$$

where

$$\alpha_1=\frac{s+\tau_o s^2}{\left[1+\beta\left(s+\tau_o s^2\right)\right]},\ \alpha_2=\frac{\alpha_1\varepsilon_1\left(1-\beta\alpha_1\right)}{\left[1-\beta\alpha_1\varepsilon_1\varepsilon_2\right]}\ \text{and}\ \alpha_3=\frac{s^2-\alpha_1\varepsilon_1\varepsilon_2\left(1-\beta\alpha_1\right)}{\left[1-\beta\alpha_1\varepsilon_1\varepsilon_2\right]}.$$

By solving the system in (29) and (30), the general solution will be as follows:

$$\bar{e} = a_1\left(\bar{\phi}_o, \bar{e}_o, k_1, k_2\right)e^{-k_1 x} + a_2\left(\bar{\phi}_o, \bar{e}_o, k_1, k_2\right)e^{-k_2 x}, \tag{31}$$

and

$$\bar{\phi} = b_1\left(\bar{\phi}_o, \bar{e}_o, k_1, k_2\right)e^{-k_1 x} + b_2\left(\bar{\phi}_o, \bar{e}_o, k_1, k_2\right)e^{-k_2 x}. \tag{32}$$

By using equations (25) and (32), we obtain

$$\bar{\theta} = \left(1-\beta k_1^2\right)b_1\left(\bar{\phi}_o, \bar{e}_o, k_1, k_2\right)e^{-k_1 x} + \left(1-\beta k_2^2\right)b_2\left(\bar{\phi}_o, \bar{e}_o, k_1, k_2\right)e^{-k_2 x}, \tag{33}$$

where $\pm k_1$ and $\pm k_2$ are the roots of the following characteristic equation

$$k^4 - (\alpha_1 + \alpha_3)k^2 + \left(\alpha_1\alpha_3 - \varepsilon_2\alpha_1\alpha_2\right) = 0. \tag{34}$$

By solving the above algebraic equation, then

$$k_1 = \sqrt{\frac{(\alpha_1+\alpha_3) + \sqrt{(\alpha_1+\alpha_3)^2 - 4\left(\alpha_1\alpha_3 - \varepsilon_2\alpha_1\alpha_2\right)}}{2}},$$

and

$$k_2 = \sqrt{\frac{(\alpha_1+\alpha_3) - \sqrt{(\alpha_1+\alpha_3)^2 - 4\left(\alpha_1\alpha_3 - \varepsilon_2\alpha_1\alpha_2\right)}}{2}}.$$

For small values of time t, this corresponds to a large value of s $\left(s \to \infty\right)$ and by using Taylor expansion; the following cases will be discussed.

4. One-Temperature Model (Lord-Shulman)

To get Lord-Shulman model, $\beta = 0$ so, the roots of the characteristic equation will be in the following form

$$k_{1,2} = \frac{s}{V_{1,2}} + Q_{1,2}\left(\varepsilon_1, \varepsilon_2, \tau_o\right) + O\left(\frac{1}{s}\right). \tag{35}$$

Moreover, the speeds of the waves are:

$$V_{1,2} = \frac{\sqrt{2}}{\sqrt{1 + \tau_o + \varepsilon_1\varepsilon_2\tau_o \pm \sqrt{\Psi}}}, \tag{36}$$

where

$$\Psi = \varepsilon_1^2\varepsilon_2^2\tau_o^2 + 2\varepsilon_1\varepsilon_2\tau_o + 2\varepsilon_1\varepsilon_2\tau_o^2 + \tau_o^2 - 2\tau_o + 1. \tag{37}$$

Equation (36) shows that the solutions have two waves propagated with speed V_1 and V_2 $\left(V_1 < V_2\right)$. V_1 is the speed propagation of the mechanical wave and V_2 is the speed propagation of the thermal wave, and the medium has no disturbance for which $x > tV_2$ with the following cases:

Case (1.1): $\tau_o \neq 0$.

When $\tau_o \neq 0$, then, V_1 and V_2 as in equation (36), in this case, the mechanical and the thermal waves propagate with finite speeds which depend on the material properties, where Lord and Shulman got that results in the generalized thermoelasticity theory [5].

Case (1.2): $\tau_o = 0$.

When $\tau_o = 0$, then, $V_1 \to 1$ and $V_2 \to \infty$, hence, only the mechanical wave propagates with finite speed, and this speed is constant and independent on the material properties, while the thermal wave propagates with infinite speed and this case is called coupled thermoelasticity or Biot's model [4].

Case (1.3): $\varepsilon_1 = 0$ and $\tau_o \neq 0$.

When $\varepsilon_1 = 0$, then, $V_1 \to 1$ and $V_2 \to \frac{1}{\sqrt{\tau_o}}$, in this case, the mechanical wave propagates with finite speed that is constant and doesn't depend on the material properties, and the thermal wave propagates with finite speed and depends on the relaxation time only which is called uncoupled thermoelasticity.

5. Classical Two-Temperature Model

To get the classical two-temperature thermoelasticity model of Youssef (15), $\beta \neq 0$ and hence, the roots of the characteristic equation of the system in (29) and (30) lead to the following cases:

Case (2.1): $\tau_o \neq 0$ or $\tau_o = 0$.

For any value of τ_o, then, $V_1 \to \sqrt{1+\varepsilon_1\varepsilon_2}$ and $V_2 \to \infty$, in this case, only the mechanical wave propagates with finite speed and depends on the material properties while the thermal wave propagates with infinite speed. Also, the two-temperature parameter β does not affect the speed of the thermal or the mechanical wave propagation.

Case (2.2): $\varepsilon_1 = 0$.

When $\varepsilon_1 = 0$, then, $V_1 \to 1$ and $V_2 \to \infty$, which is equivalent to the case (1.1). So, the classical two-temperature model of thermoelasticity Youssef (15) presents not a perfect model, where it generates the infinite speed of thermal wave propagation as the uncoupled thermoelasticity.

6. Hyperbolic Two-Temperature Generalized Thermoelasticity Theory

According to the results in case (2), another form for two-temperature thermoelasticity generates to thermal and conductive heat waves propagating with finite speed is needed.

Now, define $\ddot{\phi}$ as the acceleration of the conductive heat, and $\ddot{T}$ as the acceleration of the dynamical heat.

Assuming the difference between $\ddot{\phi}$ and $\ddot{T}$ is the proportion of the heat supplies, i.e.

$$\ddot{\phi} - \ddot{T} = c^2 \phi_{,ii}, \tag{38}$$

where c (distance/time) is constant.

Definition

"The constant c^2 is equal to the difference between the acceleration of the conductive temperature and the acceleration of the thermal temperature when the heat supply is a unit."

To apply the last equation in the above one-dimensional problem, we have to use the dimensionless in (18). Hence we have

$$\ddot{\phi} - \ddot{T} = \beta^* \frac{\partial^2 \phi}{\partial x^2}, \tag{39}$$

where $\beta^* = \frac{c^2}{c_o^2}$ is the dimensionless of the hyperbolic two-temperature parameter.

Then,

$$k^*_{1,2} = \frac{s}{V^*_{1,2}} + Q^*_{1,2}\left(\varepsilon_1, \varepsilon_2, \tau_o, \beta^*\right) + O\left(\frac{1}{s}\right) \tag{40}$$

and

$$V^*_{1,2} = \frac{\sqrt{2}\sqrt{1+\beta^*\tau_o+\varepsilon_1\varepsilon_2\beta^*\tau_o}}{\sqrt{1+\beta^*\tau_o+\varepsilon_1\varepsilon_2\tau_o+\tau_o\pm\sqrt{\Psi^*}}}, \tag{41}$$

where

$$\Psi^* = 1-2\tau_o+2\varepsilon_1\varepsilon_2\tau_o^2+\tau_o^2+\varepsilon_1^2\varepsilon_2^2\tau_o^2+2\varepsilon_1\varepsilon_2\tau_o-2\beta^*\varepsilon_1\varepsilon_2\tau_o^2-2\beta^*\tau_o^2+2\beta^*\tau_o+\beta^{*2}\tau_o^2. \tag{42}$$

Equation (41) shows that the solutions have two waves propagating with speed V_1^* and V_2^* $\left(V_1^* < V_2^*\right)$ given by (41), where V_1^* is the speed propagation of the mechanical wave, and V_2^* is the speed propagation of the thermal wave, and the medium has no disturbance for which $x > tV_2^*$ with the following cases:

Case (3.1): $\beta^* \neq 0$ and $\tau_o \neq 0$.

When $\beta^* \neq 0$ and $\tau_o \neq 0$ then, V_1^* and V_2^* as in equation (41), in this case, the mechanical and the thermal waves propagate with finite speeds which depend on the material properties, which agree with Lord and Shulman in case (1.1).

Now, the hyperbolic two-temperature parameter β^* effects on the speed of the thermal and the mechanical wave propagation.

Case (3.2): $\beta^* \neq 0$ and $\tau_o = 0$.

When $\beta^* \neq 0$ and $\tau_o = 0$, $V_1^* \to 1$ and $V_2^* \to \infty$, in this case, only the mechanical wave propagates with finite speed and this speed is constant and independent on the material properties, while the thermal wave propagates with infinite speed, and this case is equivalent to cases (1.2) and (2.2).

Case (3.3): $\beta^* \neq 0$, $\tau_o \neq 0$ and $\varepsilon_1 = 0$.

When $\beta^* \neq 0$, $\tau_o \neq 0$ and $\varepsilon_1 = 0$, then, $V_1^* \to 1$ and $V_2^* \to \sqrt{\beta^* + \frac{1}{\tau_o}}$.

In this case, the mechanical wave propagates with finite speed that is constant and doesn't depend on the material properties, and the thermal wave propagates with finite speed and depends on the relaxation time and the hyperbolic two-temperature parameter, and it agrees with the case (1.3).

Case (3.4): $\beta^* = 0$ and $\tau_o \neq 0$.

When $\beta^* = 0$ and $\tau_o \neq 0$, then, $V_1^* = V_1$ and $V_2^* = V_2$, which is equivalent to the case (1.1) and agree with Lord-Shulman results [3].

Case (3.5): $\beta^* = 0$, $\tau_o \neq 0$ and $\varepsilon_1 = 0$.

When $\beta^* = 0$, $\tau_o \neq 0$ and $\varepsilon_1 = 0$, then, $V_1 \to 1$ and $V_2 \to \frac{1}{\sqrt{\tau_o}}$, in this case, the mechanical wave propagates with finite speed is constant and doesn't depend on the material properties, while the thermal wave propagates with finite speed and depends on the relaxation time only. This case is equivalent to the case (1.3) of the uncoupled thermoelasticity.

Case (3.6): $\beta^* = 0$ and $\tau_o = 0$.

When $\beta^* = 0$ and $\tau_o = 0$, then, $V_1 \to 1$ and $V_2 \to \infty$, in this case, only the mechanical wave propagates with finite speed, and this speed is constant and independent on the material properties, while the thermal wave propagates with infinite speed and this case is equivalent to the case (1.2) or Biot model.

7. Uniqueness Theorem

Let V be an open regular region of space with boundary S occupied by the reference configuration of a homogeneous isotropic linear thermoelastic solid. S is assumed closed and bounded.

Supplement the equations of two temperature-generalized thermoelasticity (1)-(8) and (38) by prescribed boundary conditions [1]:

$$u_i = \hat{u}_i \quad \text{on } S_1 \times [0,\infty), \tag{43}$$

$$p_i = \hat{p}_i = \sigma_{ji} n_j \quad \text{on} \quad S - S_1 \times [0,\infty), \tag{44}$$

$$\phi_i = \hat{\phi}_i \quad \text{and} \quad \theta_i = \hat{\theta}_i \quad \text{on } S \times [0,\infty), \tag{45}$$

where $S_1 \subset S$ and superposed " $\hat{}$ " denotes the prescribed values on arbitrary subsets of S and their complements.

Also, the initial conditions as follows:

$$u_i = u_{i0},\ \dot{u}_i = \dot{u}_{i0},\ \varphi = \varphi_0 = \theta = \theta_0,\ \dot{\phi} = \dot{\theta} = 0 \ \text{ in } V[0,\infty) \text{ at } t = 0 \tag{46}$$

Theorem:

Given a regular region of space V+S with boundary S then there exists at most one set of single-valued functions $\sigma_{ij}(x_k,t)$ and $e_{ij}(x_k,t)$ with of $C^{(1)}$, $u_i(x_k,t)$, $\phi_i(x_k,t)$ and $T_i(x_k,t)$ of class $C^{(2)}$ in V+S, $t \ge 0$ which satisfy the equations (1)-(8) and (38) and the conditions (43)-(46) where $K, C_E, \lambda, \mu, \gamma, T_o, \rho, c$ and τ_o all are positive.

Proof:

Let there be two sets of functions $\sigma_{ij}^{(I)}$ and $\sigma_{ij}^{(II)}$, $e_{ij}^{(I)}$ and $e_{ij}^{(II)}$ …etc. and let $\tilde{\sigma}_{ij} = \sigma_{ij}^{(I)} - \sigma_{ij}^{(II)}$, $\tilde{e}_{ij} = e_{ij}^{(I)} - e_{ij}^{(II)}$, $\tilde{\phi} = \phi^{(I)} - \phi^{(II)}$ …etc.

By the linearity of the problem, it is clear that these differences also satisfy the equations mentioned above moreover, both kinematic and static boundary conditions are equal to zero (with $F_i = Q = 0$), and homogeneous counterparts of conditions (43)-(46), namely they satisfy the following field equations in $V \times (0,\infty)$:

$$\sigma_{ij,j} = \rho \ddot{u}_i \ , \tag{47}$$

$$\sigma_{ij} = 2\mu e_{ij} + (\lambda e_{kk} - \gamma \theta)\delta_{ij}, \tag{48}$$

$$q_i + \tau_o \dot{q}_i = -K \phi_{,i} \ , \tag{49}$$

$$q_{i,i} = -\rho T_0 \dot{\eta} \tag{50}$$

$$\rho T_0 \eta = \rho C_E \theta + T_0 \gamma_{ij} e_{ij} \ , \tag{51}$$

$$K\phi_{,ii} = \rho C_E (\dot{\theta} + \tau_0 \ddot{\theta}) + \gamma T_0 (\dot{e}_{kk} + \tau_o \ddot{e}_{kk}) \tag{52}$$

$$\ddot{\phi} - \ddot{\theta} = c^2 \phi_{,ii} \tag{53}$$

$$e_{ij} = ½ (u_{i,j} + u_{j,i}), \tag{54}$$

with the boundary and the initial conditions in (43)-(46).

For simplicity, the wave par has been omitted.

Now, consider the integral

$$\int_v \sigma_{ij} \dot{e}_{ij} \, dv = \int_v \sigma_{ij} \dot{u}_{i,j} \, dv = -\int_v \sigma_{ij,j} \dot{u}_i \, dv \, . \tag{59}$$

Upon inserting equation (47), the latter equation reduced to

$$\int_v (\sigma_{ij} \dot{e}_{ij} \, dv + \rho \dot{u}_i \ddot{u}_i) dv = 0 \, . \tag{60}$$

Using the equation (48), hence

$$\int_v \left[\left(2\mu e_{ij} + \lambda\delta_{ij} e_{kk} - \gamma\, \theta\delta_{ij}\right)\dot{e}_{ij} + \rho\,\dot{u}_i\ddot{u}_i\,\right] dv = 0\,. \tag{61}$$

It could be written as follows:

$$\frac{d}{dt}\int_v \left[\frac{1}{2}\lambda\, e_{kk}^2 + \mu\, e_{ij}\, e_{ij} + \frac{\rho\,\dot{u}_i\dot{u}_i}{2}\right] dv - \int_v \gamma\,\theta\,\dot{e}_{kk}\, dv = 0\,. \tag{62}$$

Substituting for $\dot{e}_{kk}$ from equation (52), then

$$T_0\frac{d}{dt}\int_v \left[\frac{1}{2}\lambda\, e_{kk}^2 + \mu\, e_{ij}\, e_{ij} + \frac{\rho\,\dot{u}_i\dot{u}_i}{2} + \frac{\rho\, C_E}{2\,T_0}\theta^2\right] dv - K\int_v \theta\,\phi_{,ii}\, dv + \tau_o\,\rho\, C_E\int_v \theta\,\ddot{\theta}\; dv + \gamma\, T_0\tau_o\int_v \theta\,\ddot{e}_{kk}\, dv = 0\,. \tag{63}$$

From the well-known inequality

$$-q_i\theta_{,i} \geq 0\,. \tag{64}$$

By using equation (49), then

$$K\int_v \theta_{,i}\,\phi_{,i}\, dv + \tau_o\int_v \theta_{,i}\,\dot{q}_i\, dv \geq 0\,, \tag{65}$$

which gives

$$K\int_v \theta_{,i}\,\phi_{,i}\, dv - \tau_o\int_v \theta\,\dot{q}_{i,i}\, dv \geq 0\,. \tag{66}$$

Inserting equations (50) and (51) in the last equation, hence

$$-K\int_v \theta\phi_{,ii}\, dv + \tau_o\,\rho\, C_E\int_v \theta\ddot{\theta}\; dv + \gamma\, T_0\tau_o\int_v \theta\;\ddot{e}_{kk}\, dv \geq 0\,. \tag{67}$$

Finally, from equations (63) and (67), we obtain

$$\frac{d}{dt}\int_v \left[\frac{1}{2}\lambda\, e_{kk}^2 + \mu\, e_{ij}\, e_{ij} + \frac{\rho\,\dot{u}_i\dot{u}_i}{2} + \frac{\rho\, C_E}{2\,T_0}\theta^2\right] dv \leq 0\,. \tag{68}$$

The integral in the left-hand side of (68) is initially zero since the difference functions satisfy homogeneous initial conditions. By inequality (68), however, this integral either decreases (or therefore becomes negative) or remains equal to zero. Since the integral is the sum of squares, only the latter alternative is possible, that is

$$\int_v \left[\frac{1}{2}\lambda\, e_{kk}^2 + \mu\, e_{ij}\, e_{ij} + \frac{\rho\,\dot{u}_i\dot{u}_i}{2} + \frac{\rho\, c_E}{2\,T_0}\theta^2\right] dv = 0, \quad t \geq 0\,. \tag{69}$$

It follows that the different functions are identically zero throughout the body and for all time this completes the proof of the theorem.

8. Numerical Application

To get the numerical result which includes the three models of thermoelasticity; one temperature of L-S; classical two-temperature model and the hyperbolic two-temperature model, and then the coefficients of the governing equations (29) and (30) will be in the form

$$\alpha_1 = \frac{s+\tau_o s^2}{\left[1+\Omega\left(s+\tau_o s^2\right)\right]}\,,\ \alpha_2 = \frac{\alpha_1\varepsilon_1\left(1-\Omega\alpha_1\right)}{\left[1-\Omega\,\alpha_1\,\varepsilon_1\varepsilon_2\right]}\ \text{and}\ \alpha_3 = \frac{s^2-\alpha_1\varepsilon_1\varepsilon_2\left(1-\Omega\alpha_1\right)}{\left[1-\Omega\,\alpha_1\,\varepsilon_1\varepsilon_2\right]}\,. \tag{72}$$

where

$$\Omega = \begin{cases} 0 & \text{for one}-\text{temperature} \\ \beta & \text{for classical two}-\text{temperature} \\ \beta^*/s^2 & \text{for hyperbolic two}-\text{temperature} \end{cases}\,. \tag{73}$$

Assume the thermal shock problem as follows:

$$e(0,t)=e_o(t)=0, \quad \phi(0,t)=\phi_o(t)=\varphi_0 H(t), \tag{74}$$

where $H(t)$ is the Heaviside unit step function and φ_0 is the thermal shock intensity?

By using Laplace transform, and then

$$\bar{e}(0,s)=\bar{e}_o(s)=0, \quad \bar{\phi}(0,s)=\bar{\phi}_o(s)=\frac{\varphi_0}{s}. \tag{75}$$

Hence, the solutions in the Laplace transform domain in the forms:

$$\bar{e}(x,s)=\frac{\alpha_2\varphi_o}{s\left(k_1^2-k_2^2\right)}\left[e^{-k_1x}-e^{-k_2x}\right], \tag{76}$$

$$\bar{\phi}(x,s)=\frac{\varphi_o}{s\left(k_1^2-k_2^2\right)}\left[\left(k_1^2-\alpha_3\right)e^{-k_1x}-\left(k_2^2-\alpha_3\right)e^{-k_2x}\right], \tag{77}$$

$$\bar{\theta}(x,s)=\frac{\varphi_o}{s\left(k_1^2-k_2^2\right)}\left[\left(1-\Omega k_1^2\right)\left(k_1^2-\alpha_3\right)e^{-k_1x}-\left(1-\Omega k_2^2\right)\left(k_2^2-\alpha_3\right)e^{-k_2x}\right], \tag{78}$$

$$\bar{\sigma}(x,s)=\frac{\varphi_o}{s\left(k_1^2-k_2^2\right)}\left[\begin{array}{l}\left(\alpha_2-\varepsilon_1\left(1-\Omega k_1^2\right)\left(k_1^2-\alpha_3\right)\right)e^{-k_1x} \\ \qquad -\left(\alpha_2-\varepsilon_1\left(1-\Omega k_2^2\right)\left(k_2^2-\alpha_3\right)\right)e^{-k_2x}\end{array}\right], \tag{79}$$

and

$$\bar{u}(x,s)=\frac{-\alpha_2\varphi_o}{sk_1k_2\left(k_1^2-k_2^2\right)}\left[k_2e^{-k_1x}-k_1e^{-k_2x}\right]. \tag{80}$$

To invert the Laplace transforms, we adopt a numerical inversion method based on a Fourier series expansion [17].

By this method, the inverse $f(t)$ of the Laplace transform $\bar{f}(s)$ is approximated by

$$f(t)=\frac{e^{vt}}{t_1}\left[\frac{1}{2}\bar{f}(v)+\mathrm{Re}\sum_{k=1}^{N}\bar{f}\left(v+\frac{ik\pi}{t_1}\right)\exp\left(\frac{ik\pi t}{t_1}\right)\right], \qquad 0<t_1<2t, \tag{81}$$

where N is a sufficiently large integer representing the number of terms in the truncated Fourier series, chosen such that

$$\exp(vt)\,\mathrm{Re}\left[\bar{f}\left(v+\frac{iN\pi}{t_1}\right)\exp\left(\frac{iN\pi t}{t_1}\right)\right]\le\varepsilon, \tag{82}$$

where ε is a prescribed small positive number that corresponds to the degree of accuracy required. The parameter v is a positive free parameter that must be greater than the real part of all the singularities of $\bar{f}(s)$. The optimal choice of v was obtained according to the criteria described in [17].

The copper material was chosen for purposes of numerical evaluations, and the constants of the problem were taken as following [1]:

$K=386$ N/K.sec, $\alpha_T=1.78(10)^{-5}\,K^{-1}$, $C_E=383.1\,m^2/K$, $\eta=8886.73$ m/sec^2

$\mu=3.86(10)^{10}$ N/m^2, $\lambda=7.76(10)^{10}$ N/m^2, $\rho=8954$ kg/m^3, $\tau_o=0.02$ sec, $T_o=293$ K,

$\varepsilon_1=1.60861$, $\varepsilon_2=0.0104442$, $\beta=0.1$ and $\beta^*=1.0$ (Assumed).

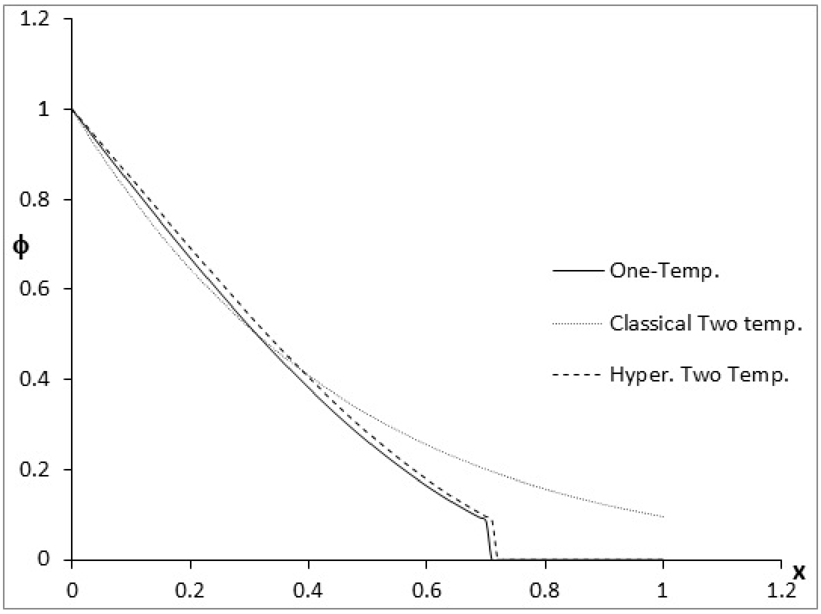

Fig. 1. The conductive temperature distribution

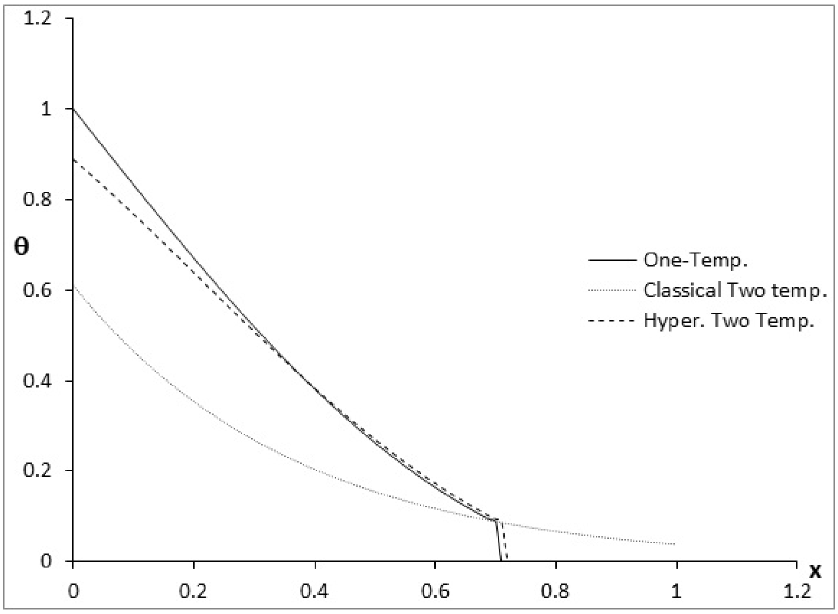

Fig. 2. The thermo-dynamical temperature distribution

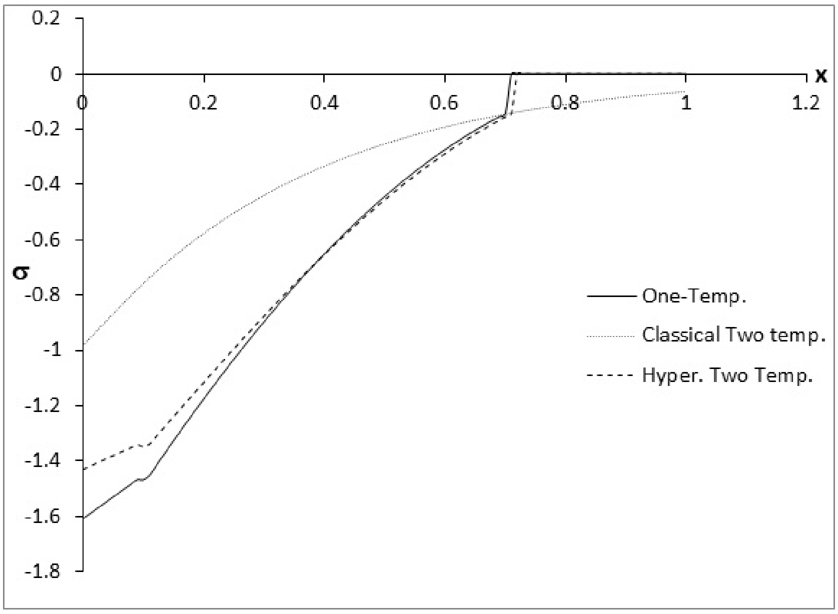

Fig. 3. The stress distribution

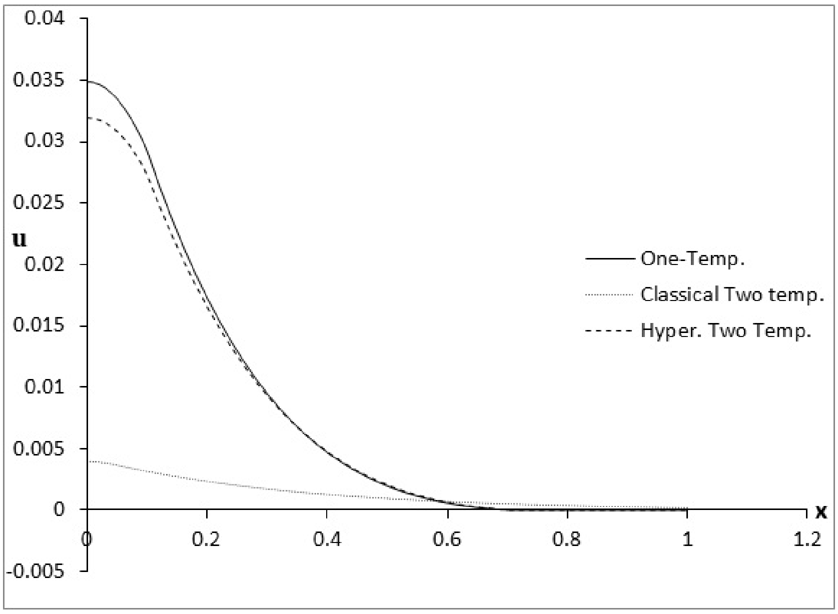

Fig. 4. The displacement distribution

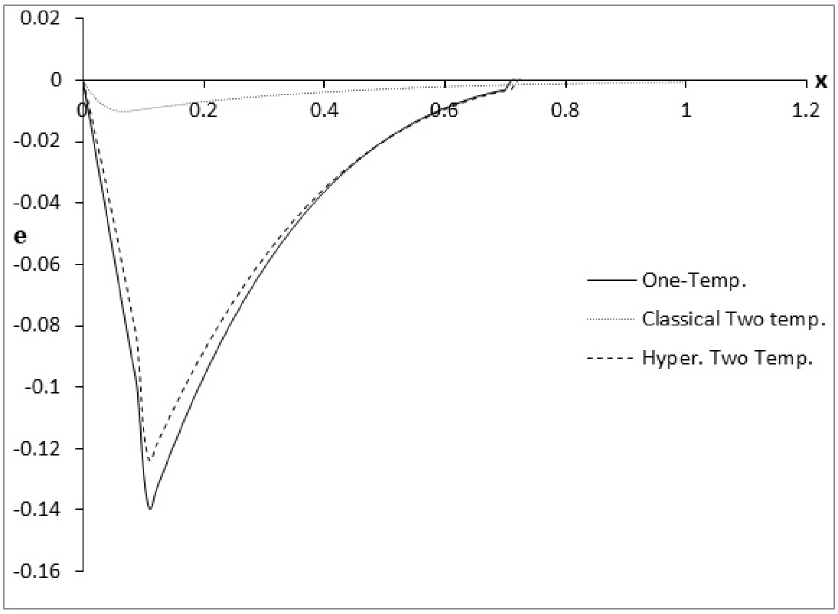

Fig. 5. The strain distribution

The Figures 1-5 show the conductive temperature, the thermo-dynamical temperature, the stress, the displacement and the strain distributions respectively for the three models; one temperature model, classical two-temperature model, and hyperbolic two-temperature model.

In Figures 1-3, the hyperbolic two-temperature model agrees with one temperature model, and they introduce finite speed of the conductive temperature, the thermo-dynamical temperature, and the stress waves propagation, while it is not in the classical two-temperature model.

In Figures 4 and 5, the hyperbolic two-temperature model agrees with one temperature model where the displacement and the strain waves vanish before the classical two-temperature model.

In Figure 5, the peak points of the strain are closed in the two cases of the one-temperature model and the hyperbolic two-temperature model, while the peak point of the classical two-temperature model has a different value and far from the others peak point.

9. Conclusion

1- The classical two-temperature generalized thermoelasticity model Youssef [1] does not introduce finite speed of the thermal wave propagation which is physically unacceptable.
2- This work introduces hyperbolic two-temperature generalized thermoelasticity model in which the thermal wave propagation has a finite speed.
3- The two-temperature parameter has significant effects on all the studied fields for the hyperbolic two-temperature generalized thermoelasticity model and the classical two-temperature generalized thermoelasticity model.
4- The numerical results of all the studied fields show that, the thermo-mechanical waves of the hyperbolic two-temperature generalized thermoelasticity model and the one-temperature generalized thermoelasticity have the same attitude.
5- The hyperbolic two-temperature generalized thermoelasticity model is a successful model to study the behavior of the thermoelastic materials.

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