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HIGHER-ORDER MODEL OF PRESTRESSED ISOTROPIC MEDIUM FOR LARGE INITIAL DEFORMATIONS

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Abstract. Within the theory of small deformations superposed on a finite one, a consistent linearization for the nonlinear equations of the mechanics of an originally isotropic elastic body in a neighborhood of some initial stress state is carried out in the Lagrange coordinate system. As the elastic potential for the originally isotropic body, we use the representation of the specific strain energy through the algebraic invariants of the Green-Lagrange strain tensor. The linearized constitutive relations and the equations of motion of the prestressed medium are derived that allow taking into account the nonlinear effects of the initial deformation on the elastic properties of the originally isotropic body.

Keywords: initial stress, prestressed, initial deformation, elastic moduli of III orders, elastic moduli of IV orders, linearized theory, large initial deformations, elastic potential of IV orders

1. Introduction

The widespread use of artificial materials in aerospace and mechanical engineering, and in electronics leads to the need for a detailed study of the physical properties of used materials, their technological and strength characteristics, depending on the operating modes and conditions. Such studies involve the solution of complex applied problems of the static and dynamic theory of elasticity, as well as the use of mathematical models that describe the processes taking place in the materials under consideration with some degree of accuracy. In turn, the modeling of these processes while taking into account various initial effects is associated with the use of an elastic potential [1, 2], which describes the energy accumulated during deformation. The choice of the particular form of the potential is determined by the specifics of the problem under consideration and by the coordinate system used. For isotropic materials, the elastic potential can be represented as a scalar function of invariants of one of the strain tensors [2-4]. Various expressions for the potential in the form of polynomials of III and IV order in invariants of the Green-Lagrange strain tensor were successfully used in [5, 6] for the modeling of highly elastic materials. For more rigid materials (rocks, metals, alloys, crystals, etc.), the Murnaghan representation of the elastic potential in the form of a cubic function of the Green-Lagrange strain tensor invariants [7] is widely used, in which, along with the elastic moduli of II order, there are also III order moduli. At present, due to theoretical and experimental studies, the values of III order moduli are known for a wide range of metals, alloys, crystals, various structural materials and for some rocks [8-12]. The use of hyperelastic material model with elastic moduli of III order made it possible to describe the properties of a prestressed medium more accurately: to analyze the second-order effects [11, 12], to determine the mechanical stresses [13, 14], and to study the features of dynamics, propagation and localization of waves [15-24]. In [21] a fairly comprehensive

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review on this subject is given. In order to develop a linearized contact theory for prestressed bodies, a consistent linearization for the nonlinear equations of the mechanics of an elastic solid has been carried out in the Lagrange and Euler coordinate systems [15]. The linearized equations of motion and the constitutive relations of the prestressed medium were derived in an arbitrary, generally curvilinear, coordinate system. The expressions presented in a compact form convenient for research were used in [15, 20-24] to solve a number of mixed boundary-value problems of the dynamic theory of elasticity. Particularly, for the model of originally isotropic hyperelastic material with the Murnaghan potential, the influence of the nature of the initial mechanical effects on the formation, propagation, and localization of wave fields in both homogeneous and inhomogeneous prestressed media has been studied [22-24].

Recently, a number of new materials, promising in practical applications and possessing unique physical properties, have appeared. In particular, these materials are able to withstand a very high level of elastic deformation, at which the nonlinearity of elastic properties becomes very significant. In [25-27], the results of experimental studies on the determination of elastic moduli of III and IV order for bulk metallic glasses based on zirconium (Zr) and palladium (Pd) are presented. The appearance of information about IV order moduli necessitate the improvement of the linearized contact theory for prestressed bodies developed in [15]. In this paper, we use a representation of the potential in which the elastic moduli of IV order are taken into account. The linearization is carried out in the Cartesian material coordinate system. The linearized equations of motion and the constitutive relations of the prestressed medium are derived, which allow taking into account the nonlinear effects of second-order and third-order in the influence of mechanical deformations on the elastic properties of the original material. Within the framework of the proposed model, we studied the effect of accounting for higher-order moduli on the parameters of the initial strained state and the properties of the prestressed material.

2. Nonlinear boundary-value problem for a prestressed originally isotropic elastic body

Consider the reference v and the actual V configurations before and after application of surface and mass forces, respectively. The position of a material point in these configurations is given by the vectors $\mathbf{r} = x_k \mathbf{i}_k$ and $\mathbf{R} = X_k \mathbf{i}_k$, where $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ is the orthonormal Cartesian vector basis, x_1, x_2, x_3 and X_1, X_2, X_3 are the Lagrangian and Eulerian coordinates. Here and below, we use the Einstein summation convention. Representation of the nabla-operators in the reference ∇_0 and the actual ∇ configurations is defined by expressions:

$$\nabla_0 = \mathbf{i}_m \frac{\partial}{\partial x_m}, \quad \nabla = \mathbf{i}_m \frac{\partial}{\partial X_m}. \tag{1}$$

Deformation of the medium is characterized by the deformation gradient $\mathbf{C}$, the Cauchy-Green strain tensor $\mathbf{G}$, and the Green-Lagrange strain tensor $\mathbf{S}$ ($\mathbf{I}$ is the unit tensor):

$$\mathbf{C} = \nabla_0 \mathbf{R}, \quad \mathbf{G} = \mathbf{C} \cdot \mathbf{C}^{\mathrm{T}}, \quad \mathbf{S} = \frac{1}{2}(\mathbf{G} - \mathbf{I}). \tag{2}$$

To describe the stress state of a medium, we use the Piola stress tensor $\mathbf{\Pi}$ and the Kirchhoff stress tensor $\mathbf{P}$, defined in the reference configuration:

$$\mathbf{\Pi} = \mathbf{P} \cdot \mathbf{C}, \quad \mathbf{P} = \boldsymbol{\chi}_{\mathrm{S}} = \frac{\partial \chi}{\partial \mathbf{S}}. \tag{3}$$

The tensor $\boldsymbol{\chi}_{\mathrm{S}}$ is the derivative of the scalar function of the elastic potential $\chi = \chi(\mathbf{S})$ with respect to the strain tensor $\mathbf{S}$. For an isotropic elastic material, the specific strain energy function can be expressed in terms of the algebraic invariants $I_k = \mathrm{tr}(\mathbf{S}^k)$ $(k = 1, 2, 3)$ of the Green-Lagrange strain tensor [1-3, 7, 15, 25]:

$$\chi = -pI_1 + \frac{1}{2}\lambda I_1^2 + \mu I_2 + \frac{1}{6}\nu_1 I_1^3 + \nu_2 I_1 I_2 + \frac{4}{3}\nu_3 I_3 + \frac{1}{24}\gamma_1 I_1^4 + \frac{1}{2}\gamma_2 I_1^2 I_2 + \frac{4}{3}\gamma_3 I_1 I_3 + \frac{1}{2}\gamma_4 I_2^2 \tag{4}$$

where λ, μ are the elastic moduli of II order; ν_1, ν_2, ν_3 and $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are the elastic moduli of III and IV orders, respectively. The moduli of III and IV orders are interpreted as linear and quadratic in deformations additives to the elastic moduli of II order, which gives one of the ways to determine them [25]. Further, when deriving the constitutive relations, we assume that the state $\mathbf{S} = 0$ is a state with a minimum free energy, $p = 0$. It should be noted that the expression (4) with the moduli of only II and III orders coincides with the Murnaghan potential [3, 7, 15].

Taking (4) into account, the Kirchhoff stress tensor (3) is written in the form:

$$\mathbf{P} = \frac{\partial \chi}{\partial I_k}\frac{\partial I_k}{\partial \mathbf{S}} = \psi_0 \mathbf{I} + 2\psi_1 \mathbf{S} + 3\psi_2 \mathbf{S}^2, \tag{5}$$

$$\psi_0 = \frac{\partial \chi}{\partial I_1} = \lambda I_1 + \frac{1}{2}\nu_1 I_1^2 + \nu_2 I_2 + \frac{1}{6}\gamma_1 I_1^3 + \gamma_2 I_1 I_2 + \frac{4}{3}\gamma_3 I_3,$$
$$\psi_1 = \frac{\partial \chi}{\partial I_2} = \mu + \nu_2 I_1 + \gamma_2 I_1^2/2 + \gamma_4 I_2, \quad \psi_2 = \frac{\partial \chi}{\partial I_3} = \frac{4}{3}\left(\nu_3 + \gamma_3 I_1\right). \tag{6}$$

Here we used the following relations for the derivatives of algebraic invariants I_k $(k = 1, 2, 3)$ with respect to the strain tensor $\mathbf{S}$ [3]:

$$\frac{\partial I_1}{\partial \mathbf{S}} = \mathbf{I}, \quad \frac{\partial I_2}{\partial \mathbf{S}} = 2\mathbf{S}, \quad \frac{\partial I_3}{\partial \mathbf{S}} = 3\mathbf{S}^2.$$

The boundary-value problem of the nonlinear elasticity for a prestressed originally isotropic body in Lagrangian coordinates is described by the equations of motion

$$\nabla_0 \cdot \mathbf{\Pi} + \rho_0 \mathbf{b} = \rho_0 \frac{\partial^2 \mathbf{R}}{\partial t^2} \tag{7}$$

and the boundary conditions on the body surface $o = o_1 + o_2$

$$o_1: \quad \mathbf{n} \cdot \mathbf{\Pi} = \mathbf{t}$$
$$o_2: \quad \mathbf{R} = \mathbf{R}^* \tag{8}$$

where $\mathbf{b}$ is the mass forces vector; $\mathbf{t}$ is the surface forces vector; $\mathbf{R}^*$ is the position vector of a point on the deformed body surface; ρ_0 is the material density in the reference configuration; and $\mathbf{n}$ is the unit vector normal to the surface of the undeformed body. The formulation of the problem is closed by the constitutive law of a hyperelastic isotropic body, which is described by the expression (5) with the coefficients (6).

3. Linearization about the initial stress state of a hyperelastic originally isotropic body

We assume that there is an initial deformed equilibrium state of the elastic body and the quantities characterizing this state do not depend explicitly on time [3, 15]:

$$\mathbf{R}_1 = \mathbf{R}_1(\mathbf{r}), \quad \mathbf{C}_1 = \nabla_0 \mathbf{R}_1, \quad \mathbf{S}_1 = \mathbf{S}(\mathbf{C}_1), \quad \mathbf{P}_1 = \mathbf{P}(\mathbf{C}_1), \quad \mathbf{\Pi}_1 = \mathbf{\Pi}(\mathbf{C}_1). \tag{9}$$

Equilibrium equations in the volume and on the surface $o = o_1 + o_2$ in the basis of the reference configuration are given by the relations:

$$\nabla_0 \cdot \mathbf{\Pi}_1 + \rho_0 \mathbf{b}_1 = 0, \tag{10}$$

$$o_1: \quad \mathbf{n}\cdot\mathbf{\Pi}_1 = \mathbf{t}_1,$$
$$o_2: \quad \mathbf{R}_1 = \mathbf{R}_1^*. \tag{11}$$

Consider a small perturbation of the initial deformed equilibrium state (9), caused by a small change in the mass or surface forces (characteristics of the perturbed state will be denoted by the superscript $^\times$):

$$\mathbf{b}^\times = \mathbf{b}_1 + \eta\mathbf{d}, \quad \mathbf{t}^\times = \mathbf{t}_1 + \eta\mathbf{f}. \tag{12}$$

Then the position of the points in the perturbed state of the medium is determined by the vector

$$\mathbf{R}^\times = \mathbf{R}_1 + \eta\mathbf{u}. \tag{13}$$

Here η is the small parameter; $\mathbf{u}$ is the vector of additional displacements.

For the Piola stress tensor, the following representation holds in the perturbed state:

$$\mathbf{\Pi}^\times = \mathbf{\Pi}_1 + \eta\mathbf{\Pi}^\bullet + \eta^2(...) + \cdots. \tag{14}$$

By retaining in the expansion only the terms linear in η, we obtain

$$\mathbf{\Pi}^\bullet = \frac{d}{d\eta}\mathbf{\Pi}\left(\mathbf{C}_1 + \eta\nabla_0\mathbf{u}\right)\Big|_{\eta=0}. \tag{15}$$

Here and below, the superscript $^\bullet$ denotes the convective derivatives of the corresponding tensors.

The quantities (12) – (14), which determine the perturbed state of the body, must satisfy the equations of motion (7) and the boundary conditions (8). Taking into account the equilibrium of the stress-strain state (9) – (11) and the expressions (12) – (15), we derive the linearized (up to accuracy $o\left(\eta^2\right)$) equations of motion in the absence of mass forces:

$$\nabla_0\cdot\mathbf{\Pi}^\bullet = \rho_0\frac{\partial^2\mathbf{u}}{\partial t^2} \tag{16}$$

and the linearized boundary conditions on the surface $o = o_1 + o_2$:

$$o_1: \quad \mathbf{n}\cdot\mathbf{\Pi}^\bullet = \mathbf{f}$$
$$o_2: \quad \mathbf{u} = \mathbf{u}^* \tag{17}$$

To find the convective derivative of the Piola stress tensor, its representation in terms of the Kirchhoff stress tensor (3) is used:

$$\mathbf{\Pi}^\bullet = \mathbf{P}^\bullet\cdot\mathbf{C}_1 + \mathbf{P}_1\cdot\mathbf{C}^\bullet. \tag{18}$$

Taking (5), (6) into account, for the convective derivative of the Kirchhoff stress tensor, we obtain:

$$\mathbf{P}^\bullet = \psi_0^\bullet\mathbf{I} + 2\psi_1^\bullet\mathbf{S}_1 + 2\psi_1\mathbf{S}^\bullet + 3\psi_2^\bullet\mathbf{S}_1^2 + 3\psi_2\left(\mathbf{S}_1\cdot\mathbf{S}^\bullet + \mathbf{S}^\bullet\cdot\mathbf{S}_1\right), \tag{19}$$

where

$$\psi_k^\bullet = \psi_{km}I_m^\bullet, \quad \psi_{km} = \frac{\partial^2\chi}{\partial I_{k+1}\partial I_m}, \quad I_m^\bullet = \frac{\partial I_m}{\partial\mathbf{S}}\circ\mathbf{S}^\bullet, \quad k = 0,1,2, \quad m = 1,2,3,$$
$$I_1^\bullet = \mathbf{I}\circ\mathbf{S}^\bullet, \quad I_2^\bullet = 2\mathbf{S}_1\circ\mathbf{S}^\bullet, \quad I_3^\bullet = 3\mathbf{S}_1^2\circ\mathbf{S}^\bullet, \tag{20}$$
$$\mathbf{S}^\bullet = \frac{1}{2}\left(\mathbf{C}_1\cdot\mathbf{C}^{\mathrm{T}\bullet} + \mathbf{C}^\bullet\cdot\mathbf{C}_1^{\mathrm{T}}\right), \quad \mathbf{C}^\bullet = \nabla_0\mathbf{u}.$$

Here $\circ$ denotes the full product.

Further it is assumed that the initial stressed state in the originally isotropic hyperelastic body is due to the uniform initial deformation:

$$\mathbf{R}_1 = \mathbf{r}\cdot\mathbf{\Lambda}, \quad \mathbf{\Lambda} = \delta_{km}\nu_k\mathbf{i}_k\otimes\mathbf{i}_m; \quad \nu_k = \mathrm{const}, \quad k = 1,2,3, \tag{21}$$

where v_1, v_2, v_3 are the stretch ratios along the Cartesian coordinate axes; δ_{km} is the Kronecker delta.

Within the framework of assumptions (21), it follows from (1), (2) that the deformation gradient, the Cauchy-Green strain tensor, and the Green-Lagrange strain tensor have the form:

$$\mathbf{C}_1 = v_k \mathbf{i}_k \otimes \mathbf{i}_k,\ \mathbf{G}_1 = v_k^2 \mathbf{i}_k \otimes \mathbf{i}_k,\ \mathbf{S}_1 = S_k \mathbf{i}_k \otimes \mathbf{i}_k;\quad S_k = \frac{1}{2}\left(v_k^2 - 1\right),\ k = 1,2,3. \tag{22}$$

The Piola and Kirchhoff stress tensors are expressed as follows:

$$\mathbf{\Pi}_1 = v_k P_k \mathbf{i}_k \otimes \mathbf{i}_k,\quad \mathbf{P}_1 = P_k \mathbf{i}_k \otimes \mathbf{i}_k;\ P_k = \psi_0 + 2\psi_1 S_k + 3\psi_2 S_k^2,\ k = 1,2,3. \tag{23}$$

Taking (4), (6) into account, the convective derivatives of the stress tensors (18), (19) take the form:

$$\mathbf{\Pi}^\bullet = \Pi_{km}^* \mathbf{i}_k \otimes \mathbf{i}_m,\qquad \mathbf{P}^\bullet = P_{km}^* \mathbf{i}_k \otimes \mathbf{i}_m;$$

$$\Pi_{km}^* = \delta_{km} v_m v_n \xi_{kn} \frac{\partial u_n}{\partial x_n} + v_m \left[\psi_1 + \frac{3}{2}\psi_2\left(S_k + S_m\right)\right]\left(v_k \frac{\partial u_k}{\partial x_m} + v_m \frac{\partial u_m}{\partial x_k}\right) + P_k \frac{\partial u_m}{\partial x_k},$$

$$P_{km}^* = \delta_{km} v_n \xi_{kn} \frac{\partial u_n}{\partial x_n} + \left[\psi_1 + \frac{3}{2}\psi_2\left(S_k + S_m\right)\right]\left(v_k \frac{\partial u_k}{\partial x_m} + v_m \frac{\partial u_m}{\partial x_k}\right),\qquad k,m = 1,2,3; \tag{24}$$

$$\xi_{kn} = \psi_{01} + 2\psi_{02} S_n + 3\psi_{03} S_n^2 + 2\left(\psi_{11} + 2\psi_{12} S_n\right) S_k + 4\psi_{21} S_k^2,\qquad k,n = 1,2,3.$$

Here we used the following relations

$$\mathbf{S}^\bullet = \frac{1}{2}\left(v_k \frac{\partial u_k}{\partial x_m} + v_m \frac{\partial u_m}{\partial x_k}\right)\mathbf{i}_k \otimes \mathbf{i}_m,$$

$$I_1^\bullet = v_k \frac{\partial u_k}{\partial x_k},\quad I_2^\bullet = 2 v_k S_k \frac{\partial u_k}{\partial x_k},\quad I_3^\bullet = 3 v_k S_k^2 \frac{\partial u_k}{\partial x_k},$$

$$\psi_0^\bullet = \left(\psi_{01} + 2\psi_{02} S_n + 3\psi_{03} S_n^2\right) v_n \frac{\partial u_n}{\partial x_n},$$

$$\psi_1^\bullet = \left(\psi_{11} + 2\psi_{12} S_n\right) v_n \frac{\partial u_n}{\partial x_n},\quad \psi_2^\bullet = \psi_{21} v_n \frac{\partial u_n}{\partial x_n}.$$

The coefficients ψ_{km} defined in (20), according to (4), (21), have the form:

$$\psi_{01} = \lambda + \nu_1 I_1 + \frac{1}{2}\gamma_1 I_1^2 + \gamma_2 I_2,\quad \psi_{02} = \psi_{11} = \nu_2 + \gamma_2 I_1,$$

$$\psi_{03} = \psi_{21} = \frac{4}{3}\gamma_3,\quad \psi_{12} = \gamma_4,\quad \psi_{13} = \psi_{22} = \psi_{23} = 0, \tag{25}$$

$$I_m = S_1^m + S_2^m + S_3^m,\quad m = 1,2,3.$$

Using the formulae (6), (24), (25), the linearized equations (16) can be written in the scalar form ($m = 1,2,3$):

$$v_m v_n \xi_{mn} \frac{\partial^2 u_n}{\partial x_m \partial x_n} + v_m \left[\psi_1 + \frac{3}{2}\psi_2\left(S_k + S_m\right)\right]\left(v_k \frac{\partial^2 u_k}{\partial x_k \partial x_m} + v_m \frac{\partial^2 u_m}{\partial x_k^2}\right) + P_k \frac{\partial^2 u_m}{\partial x_k^2} = \rho_0 \frac{\partial^2 u_m}{\partial t^2}. \tag{26}$$

The expressions (24) for the components of the linearized Piola stress tensor $\mathbf{\Pi}^\bullet$ can be represented in a more compact and traditional for anisotropic materials form [2, 3, 15, 22]:

$$\Pi_{ij}^* = C_{ijkl}^* \frac{\partial u_k}{\partial x_l},\qquad i, j = 1,2,3, \tag{27}$$

where

$$C^*_{iiii} = v_i^2 \xi_{ii} + 2v_i^2 \left(\psi_1 + 3\psi_2 S_i\right) + P_i, \quad C^*_{iijj} = v_i v_j \xi_{ij}, \qquad i \neq j,$$
$$C^*_{ijij} = v_j v_i \left(\psi_1 + \frac{3}{2}\psi_2 \left(S_i + S_j\right)\right), \quad C^*_{ijji} = v_j^2 \left(\psi_1 + \frac{3}{2}\psi_2 \left(S_i + S_j\right)\right) + P_i, \tag{28}$$

and all the remaining coefficients C^*_{ijkl} are zero.

The relations (28) show the influence of the initial deformations on the original properties of isotropic material, i.e. C^*_{ijkl} determine the properties of the prestressed material. According to (6), (22) – (25), in the absence of initial stresses ($v_1 = v_2 = v_3 = 1$) we have:

$$C^*_{iiii} = \lambda + 2\mu, \quad C^*_{iijj} = \lambda, \quad C^*_{ijij} = \mu, \quad C^*_{ijji} = \mu .$$

Taking (27) into account, the linearized equations of motion (26) for a prestressed originally isotropic body take the form [1-3, 15, 22]:

$$C^*_{ijkl} \frac{\partial^2 u_k}{\partial x_l \partial x_i} = \rho_0 \frac{\partial^2 u_j}{\partial t^2}, \quad j = 1, 2, 3. \tag{29}$$

When solving boundary-value problems for semi-bounded media, the problem (17), (29) is closed by additional boundary conditions, depending on the medium type:

– for the half-space $|x_1|, |x_2| \leq \infty, \ x_3 \leq 0$

$$x_3 \to -\infty: \ \mathbf{u} \to 0; \tag{30}$$

– for the layer $|x_1|, |x_2| \leq \infty, \ x_0 \leq x_3 \leq x_{30}$, the lower bound of which is rigidly clamped

$$x_3 = x_0: \ \mathbf{u} = 0; \tag{31}$$

– at the interface of n-th and $n+1$-th structural elements with plane-parallel boundaries (with full coupling):

$$x_3 = x_0: \ \mathbf{u}^{(n)} = \mathbf{u}^{(n+1)}, \quad \Pi^{*(n)}_{3k} = \Pi^{*(n+1)}_{3k}, \quad k = 1, 2, 3; \tag{32}$$

– at the interface (in contact without friction):

$$x_3 = x_0: \ u_3^{(n)} = u_3^{(n+1)} = 0, \quad \Pi^{*(n)}_{3k} = \Pi^{*(n+1)}_{3k} = 0, \quad k = 1, 2. \tag{33}$$

4. Determination of the initial stress state

The prestressed state of a body is described by the Kirchhoff stress tensor, which, for the uniform initial deformation (21), is defined in (23) while taking into account the elastic moduli of II, III and IV orders. By grouping the terms in powers of the deformation tensor $\mathbf{S}$, for the components of the Kirchhoff stress tensor we obtain:

$$P_1 = \left(\lambda + 2\mu\right) S_1 + \lambda S_2 + \lambda S_3 + H_1^2 + H_1^3,$$
$$P_2 = \lambda S_1 + \left(\lambda + 2\mu\right) S_2 + \lambda S_3 + H_2^2 + H_2^3, \tag{34}$$
$$P_2 = \lambda S_1 + \lambda S_2 + \left(\lambda + 2\mu\right) S_3 + H_3^2 + H_3^3.$$

If in the representation (4) only the terms with the elastic moduli of II and III orders are retained, then for $\psi_k \ \left(k = 1, 2, 3\right)$, according to (6), we have:

$$\psi_0 = \lambda I_1 + \frac{1}{2}\nu_1 I_1^2 + \nu_2 I_2, \quad \psi_1 = \mu + \nu_2 I_1, \quad \psi_2 = \frac{4}{3}\nu_3,$$

and $H_k^3 = 0$ in the expressions (34).

If the elastic potential (4) has only the terms with the moduli of II order, then the coefficients $\psi_k \ \left(k = 1, 2, 3\right)$ are determined by the relations:

$$\psi_0 = \lambda I_1, \quad \psi_1 = \mu, \quad \psi_2 = 0, \quad H_k^2 = H_k^3 = 0 .$$

The components of the Kirchhoff stress tensor in this case are linearly related to the components of the strain tensor.

The terms H_k^2 and H_k^3 $(k=1,2,3)$ from (34) allow one to take into account the influence of the elastic moduli of III and IV orders:

$$H_1^2=(a_1+2a_2)S_1^2+2a_1S_1S_2+2a_1S_1S_3+a_1S_2^2+a_1S_3^2+\nu_1S_2S_3,$$
$$H_2^2=(a_1+2a_2)S_2^2+2a_1S_2S_1+2a_1S_2S_3+a_1S_1^2+a_1S_3^2+\nu_1S_1S_3,$$
$$H_3^2=(a_1+2a_2)S_3^2+2a_1S_3S_2+2a_1S_3S_1+a_1S_1^2+a_1S_2^2+\nu_1S_1S_2, \quad (35)$$
$$a_1=\frac{1}{2}\nu_1+\nu_2,\ a_2=\nu_2+2\nu_3.$$

$$H_1^3=\frac{1}{3}(b_1+b_2+4b_4)S_1^3+(b_1+2b_3)(S_2+S_3)S_1^2+\left(2b_1S_2S_3+(b_1+b_2)\left(S_2^2+S_3^2\right)\right)S_1+$$
$$+\frac{1}{3}(b_1+2b_3)\left(S_2^3+S_3^3\right)+b_1S_2S_3(S_2+S_3),$$
$$H_2^3=\frac{1}{3}(b_1+b_2+4b_4)S_2^3+(b_1+2b_3)(S_1+S_3)S_2^2+\left(2b_1S_1S_3+(b_1+b_2)\left(S_1^2+S_3^2\right)\right)S_2+ \quad (36)$$
$$+\frac{1}{3}(b_1+2b_3)\left(S_1^3+S_3^3\right)+b_1S_1S_3(S_1+S_3),$$
$$H_3^3=\frac{1}{3}(b_1+b_2+4b_4)S_3^3+(b_1+2b_3)(S_1+S_2)S_3^2+\left(2b_1S_1S_2+(b_1+b_2)\left(S_1^2+S_2^2\right)\right)S_3+$$
$$+\frac{1}{3}(b_1+2b_3)\left(S_1^3+S_3^3\right)+b_1S_1S_2(S_1+S_2),$$
$$b_1=\frac{1}{2}\gamma_1+\gamma_2,\quad b_2=\gamma_2+2\gamma_4,\quad b_3=\gamma_2+2\gamma_3,\quad b_4=\gamma_2+\gamma_4+4\gamma_3.$$

Relations (34) represent a system of three generally nonlinear equations with respect to three unknowns: S_1,S_2,S_3 or P_1,P_2,P_3 or their combinations, depending on the method for specifying the initial stress-strain state. With the combined specification of the initial state, the deformation along one axis (for example, v_1) is assumed to be given, as well as two conditions for the stresses. The deformations and stresses along the other axes are determined from the system (34).

Further we studied the influence of $(i,j,k=1,2,3;\ i\neq j\neq k)$

uniaxial $1x_i \Rightarrow \quad P_i=P,\quad P_j=P_k=0$,

biaxial $2x_i \Rightarrow \quad P_i=0,\quad P_j=P_k=P$,

triaxial $3x_i \Rightarrow \quad P_i=P,\quad P_j=P_k=G$,

and hydrostatic $3x \Rightarrow \quad P_1=P_2=P_3=P$ initial loadings.

The material used is the metallic glass $Pd_{40}Cu_{30}Ni_{10}P_{20}$ with the following parameters [27]:

$$\rho=9300\,kg/m^3,\qquad \lambda=1.453\cdot10^{11}\,Pa,\qquad \mu=0.358\cdot10^{11}\,Pa,$$
$$\nu_1=-2.27\cdot10^{11}\,Pa,\qquad \nu_2=-2.34\cdot10^{11}\,Pa,\qquad \nu_3=-0.818\cdot10^{11}\,Pa,$$
$$\gamma_1=-105{,}828\cdot10^{12}\,Pa,\qquad \gamma_2=15.556\cdot10^{11}\,Pa,$$
$$\gamma_3=1.81\cdot10^{11}\,Pa,\qquad \gamma_4=-2.98\cdot10^{11}\,Pa.$$

The results of the numerical analysis are presented in the dimensionless parameters. The elastic coefficients and initial stresses are related to the shear modulus μ of the isotropic material in the reference state.

Figs. 1 and 2 show the dependences of the initial stresses on deformations obtained from the solution of the system (34), while taking into account the elastic moduli of II order only ($H_k^2 = H_k^3 = 0$, $k = 1,2,3$, dotted lines, Curves 1), the moduli of III order ($H_k^2 \neq 0$, $H_k^3 = 0$, dashed lines, Curves 2), and both the elastic moduli of III and IV orders ($H_k^2 \neq 0$, $H_k^3 \neq 0$, solid lines, Curves 3). Fig. 1 shows the stresses for uniaxial ($1x_i$), biaxial ($2x_i$), and hydrostatic ($3x$) states, respectively.

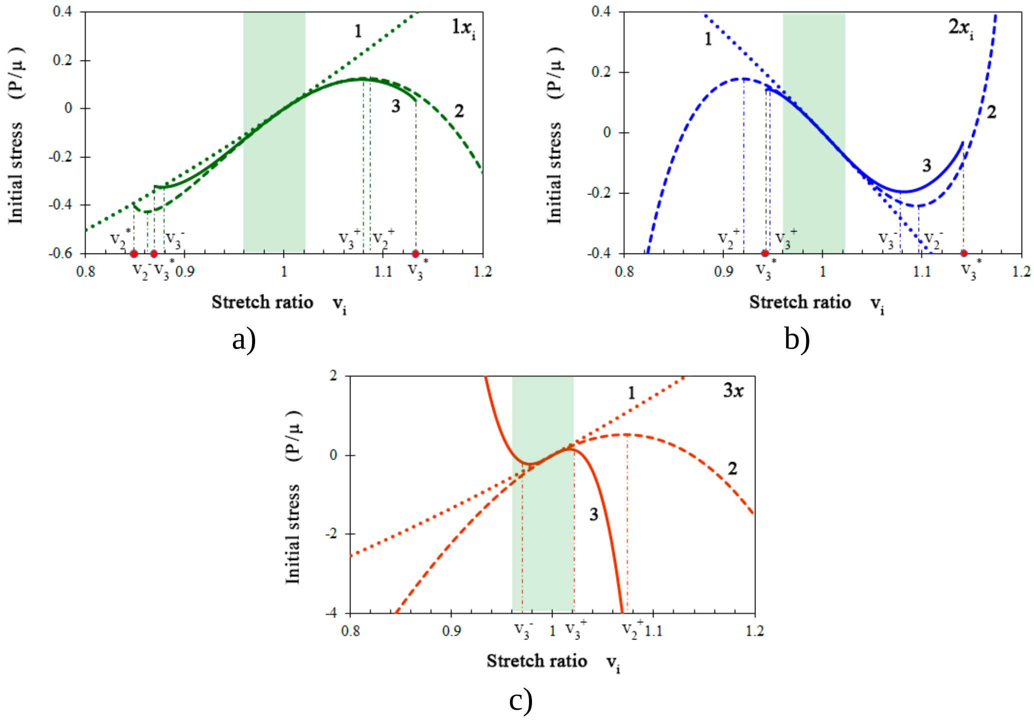

Fig. 1. The effect of accounting for higher-order moduli on stresses in the case of uniaxial (a), biaxial (b) and hydrostatic (c) loadings

Fig. 2 shows the initial stresses for triaxial ($3x_i$) states at $G/\mu = 0.1$ (Fig. 2a) and $G/\mu = -0.1$ (Fig. 2b).

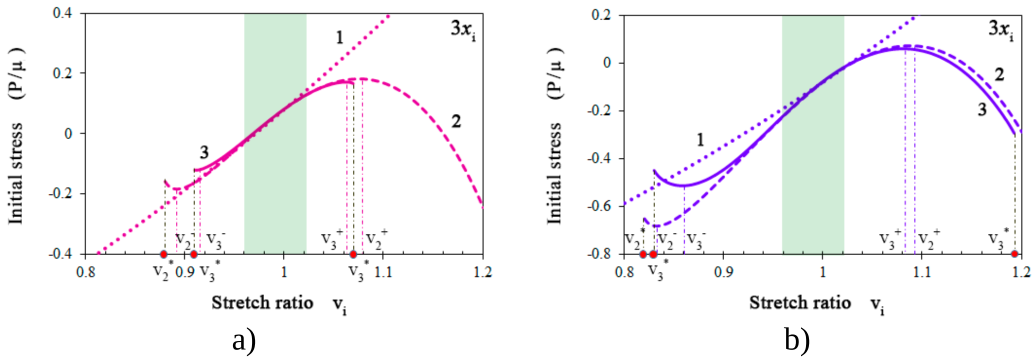

Fig. 2. The effect of accounting for higher-order moduli on stresses in the case of triaxial loading: G/μ= 0.1 (a), G/μ= –0.1 (b)

In the Figs. 1 and 2, the critical values of the stretch ratio v_i $(i = 1,2,3)$, corresponding to the Curve k $(k = 2,3)$, at which the initial stresses become complex-valued are denoted by

v_k^*, and the values at which the stresses reach a local minimum or maximum – by v_k^- and v_k^+, respectively. In the range $\left[v_k^-, v_k^+\right]$, the condition of a one-to-one correspondence is fulfilled. The shaded area is the region of physical linearity where the effect of accounting for the elastic moduli of III and IV orders is insignificant. It should be noted that the region corresponding to the uniaxial stress state was selected. From the Figs. it is clear that generally the linearity region depends on the type of initial loading: for hydrostatic (Fig. 1c) and various triaxial loadings (Fig. 2b), it is significantly reduced and can be shifted toward compression or extension.

5. Accounting for the effect of elastic moduli of higher-order on the properties of prestressed originally isotropic bodies

The role of the stress tensor in prestressed elastic body plays the linearized Piola stress tensor $\mathbf{\Pi}^\bullet$ (27) involved in the equations of motion (29) with boundary conditions (30) – (33), depending on the problem and the medium type. The influence of initial stresses on the properties of originally isotropic material is represented by the coefficients C_{ijkl}^* from (28).

We rewrite C_{ijkl}^* in the form [15, 22]:

$$C_{ijkl}^* = P_i \delta_{jk} + v_j v_k C_{ijkl}^\times, \quad i, j, k, l = 1, 2, 3. \tag{37}$$

Here $C_{ijkl}^\times$ depend on the material properties and the type of initial loading

$$\begin{aligned}
C_{iiii}^\times &= \lambda + 2\mu + 4a_2 S_i + 2a_1 I_1 + 4\left(\frac{7}{3}\gamma_3 + \gamma_4\right) S_i^2 + 4b_3 I_1 S_i + b_1 I_1^2 + b_2 I_2, \\
C_{iijj}^\times &= \lambda + 2\nu_2 \left(S_i + S_j\right) + \nu_1 I_1 + \frac{4}{3}\gamma_3 \left(4S_i^2 + 3S_j^2\right) + 2\gamma_2 \left(S_i + S_j\right) I_1 + 4\gamma_4 S_i S_j + \gamma_2 I_2 + \frac{1}{2}\gamma_1 I_1^2, \\
C_{ijij}^\times &= C_{ijji}^\times = \mu + 2\nu_3 \left(S_i + S_j\right) + \nu_2 I_1 + 2\gamma_3 I_1 \left(S_i + S_j\right) + \gamma_4 I_2 + \frac{1}{2}\gamma_2 I_1^2, \quad i, j = 1, 2, 3, \quad i \neq j
\end{aligned} \tag{38}$$

All the remaining coefficients $C_{ijkl}^\times$ are zero; the parameters a_m, b_m $\left(m = 1, 2, 3\right)$ are defined in (35), (36).

Figs. 3 and 4 show the influence of deformation on four types of elastic coefficients in a prestressed body: C_{iiii}^*, C_{iijj}^* (Fig. 3) and C_{ijij}^*, C_{ijji}^* (Fig. 4) for the uniaxial ($1x_1$) and triaxial ($3x_1$, $G/\mu = -0.1$) initial loadings. As before, the approximations of C_{ijkl}^* which take into account the elastic moduli of II, II and III, and also II, III and IV orders are indicated by dotted lines (Curves 1, 1'), dashed lines (Curves 2, 2') and solid lines (Curves 3, 3'), respectively. The numbers 1,2,3 and 1',2',3' in Fig. 3a,b mark the approximations of C_{1111}^* and C_{2222}^*, in Fig. 3c,d – the approximations of C_{1122}^* and C_{2233}^*, in Fig. 4a,b – the approximations of C_{2323}^* and C_{2332}^*, and in Fig. 4c,d – the approximations of C_{1313}^* and C_{1331}^*, respectively.

As can be seen from Figs. 3 and 4, even in the region of linearity for the initial stresses (the shaded area), the account of higher-order moduli in a prestressed body leads to significant changes in the behavior of C_{ijkl}^*. So the intersection points of the approximations of C_{1111}^* and C_{2222}^*, C_{1122}^* and C_{2233}^* for $1x_1$ correspond to the values of the coefficients in the reference state ($v_1 = 1$, Fig. 3a,c). In the case of triaxial stress state ($3x_1$, $G/\mu = -0.1$), the intersection of C_{1111}^* and C_{2222}^*, C_{1122}^* and C_{2233}^* occurs in a certain deformed state ($v_1 \neq 1$, Fig. 3b,d), the value of the coefficients at the point of intersection depends both on the type of this state and on the constants involved in the corresponding approximations.

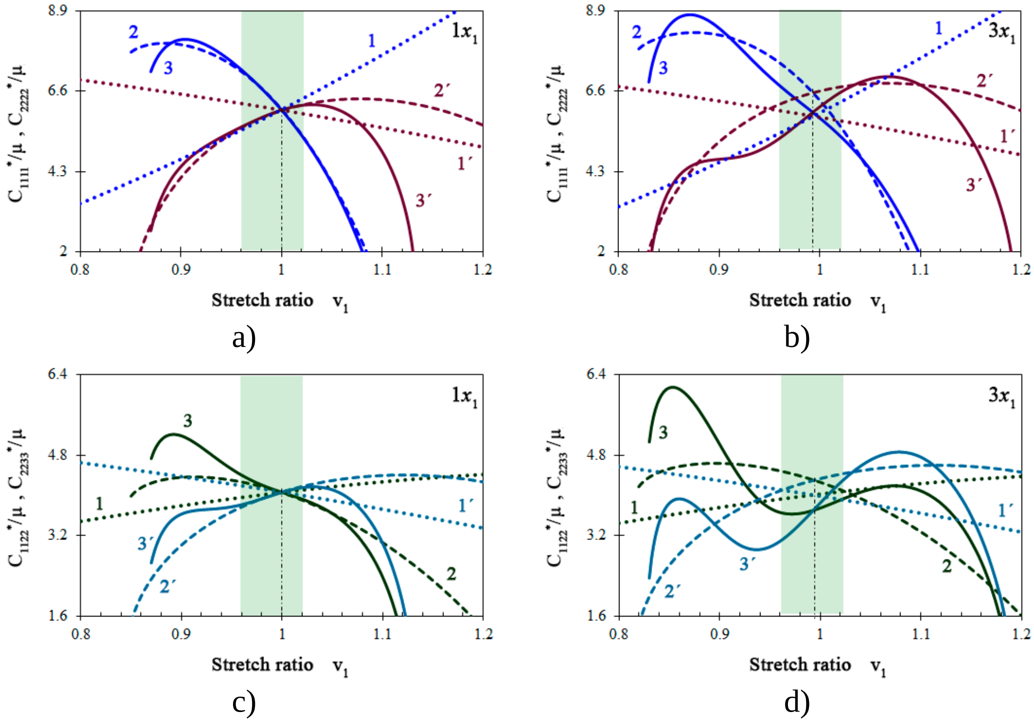

Fig. 3. The effect of accounting for higher-order moduli of original material on the coefficients C^{*}_{iiii}, C^{*}_{iijj} in the case of uniaxial (a, c) and triaxial (b, d) initial loadings

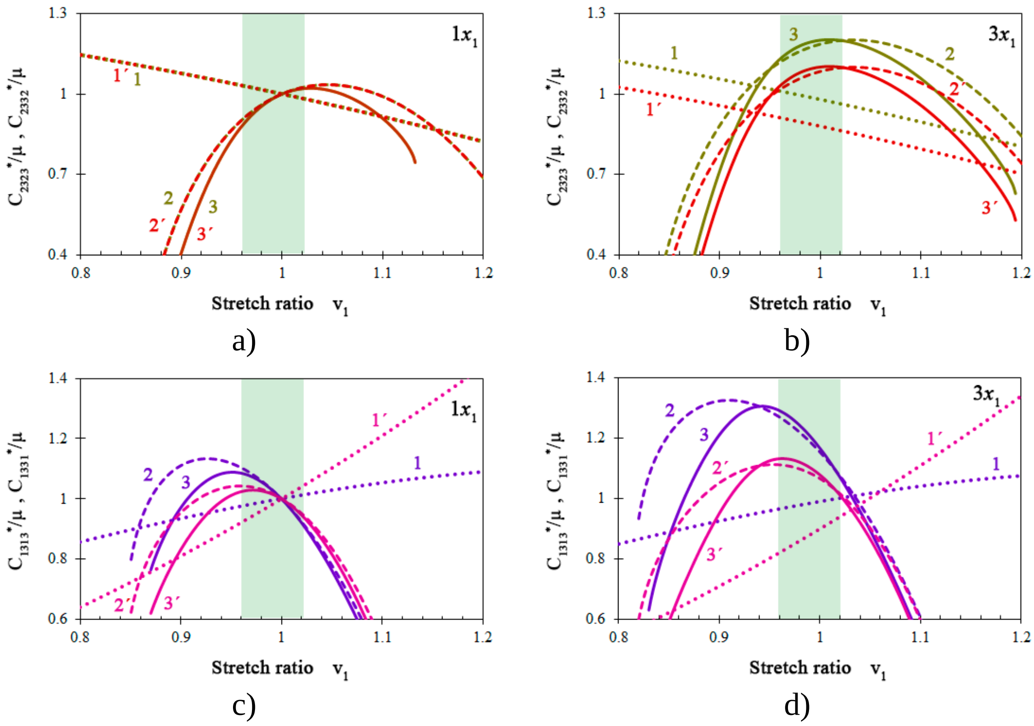

Fig. 4. The effect of accounting for higher-order moduli of original material on the coefficients C^{*}_{ijij}, C^{*}_{ijji} in the case of uniaxial (a, c) and triaxial (b, d) initial loadings

It should be noted that if for uniaxial initial loadings in the shaded area there are significant differences in the behavior of the approximations of C^{*}_{iiii}, C^{*}_{iijj}, which take into account either only Lame moduli or higher-order moduli as a whole (the differences between the approximations taking into account the moduli of III order, and III and IV orders in the shaded area are negligible), then for more complex triaxial loadings, the differences due to taking into account the IV order moduli become more substantial. In the case of a uniaxial initial deformed state for the coefficients C^*_{ijij}, C^*_{ijji}, as for C^*_{iiii}, C^*_{iijj}, it is typical that there is a point of intersection of all approximations (Fig. 4a,c), but for $3x_1$ $(G/\mu = -0.1)$ there is no such point.

6. Linear deformation approach

In the case of small initial deformation, a linear approximation is used for the invariants of the strain tensor:

$v_k = 1 + \delta_k, \quad v_k{}^2 = 1 + 2\delta_k, \quad v_k v_i = 1 + \delta_k + \delta_i, \quad i, k = 1, 2, 3, \quad i \neq k,$

$S_k = \delta_k, \quad I_1 = \theta = \delta_1 + \delta_2 + \delta_3, \quad I_2 = I_3 = 0,$

where $\delta_1, \delta_2, \delta_3$ are the relative axial compressions/tensions.

According to (6), (25), the coefficients ψ_k and ψ_{km} $\left(k = 1, 2, 3, m = 0, 1, 2\right)$ in the linear approximation take the simple form:

$$\psi_0 = \lambda\theta, \quad \psi_1 = \mu + \nu_2\theta, \quad \psi_2 = \frac{4}{3}\left(\nu_3 + \gamma_3\theta\right),$$

$$\psi_{01} = \lambda + \nu_1\theta, \quad \psi_{02} = \psi_{11} = \nu_2 + \gamma_2\theta, \quad \psi_{03} = \psi_{21} = \frac{4}{3}\gamma_3,$$

$$\psi_{12} = \gamma_4, \quad \psi_{13} = \psi_{22} = \psi_{23} = 0.$$

The components of the Kirchhoff stress tensor $\mathbf{P}$ then are expressed as follows:

$P_k = \lambda\theta + 2\mu\delta_k, \quad k = 1, 2, 3.$

The coefficients C^*_{ijkl} from the representation (27), (28) of the components of the linearized Piola stress tensor $\mathbf{\Pi}^\bullet$ in this case can be written in the form:

$$\begin{aligned}
C^*_{iiii} &= P_i + \left(\lambda + 2\mu\right) + \nu_1\theta + 2\left(\lambda + 2\mu + 2\nu_2\right)\delta_i, \\
C^*_{iijj} &= \lambda + \nu_1\theta + \left(\lambda + 2\nu_2\right)\left(\delta_i + \delta_j\right), \\
C^*_{ijij} &= \mu + \left(\mu + 2\nu_3\right)\left(\delta_i + \delta_j\right), \\
C^*_{ijji} &= P_i + \mu + 2\mu\delta_j + 2\nu_3\left(\delta_i + \delta_j\right), \qquad i, j = 1, 2, 3, \; i \neq j.
\end{aligned} \tag{39}$$

Fig. 5 shows the effect of accounting for the nonlinearity of deformation on various properties of prestressed material in the case of uniaxial ($1x_1$) initial loading. As in Figs. 3 and 4, the approximations of the coefficients C^*_{ijkl}, calculated while taking into account the elastic moduli of II order (Curves 1), II and III orders (Curves 2), and II, III and IV orders (Curves 3), are presented. The number 0 indicates the approximations (39) linear with respect to initial deformation.

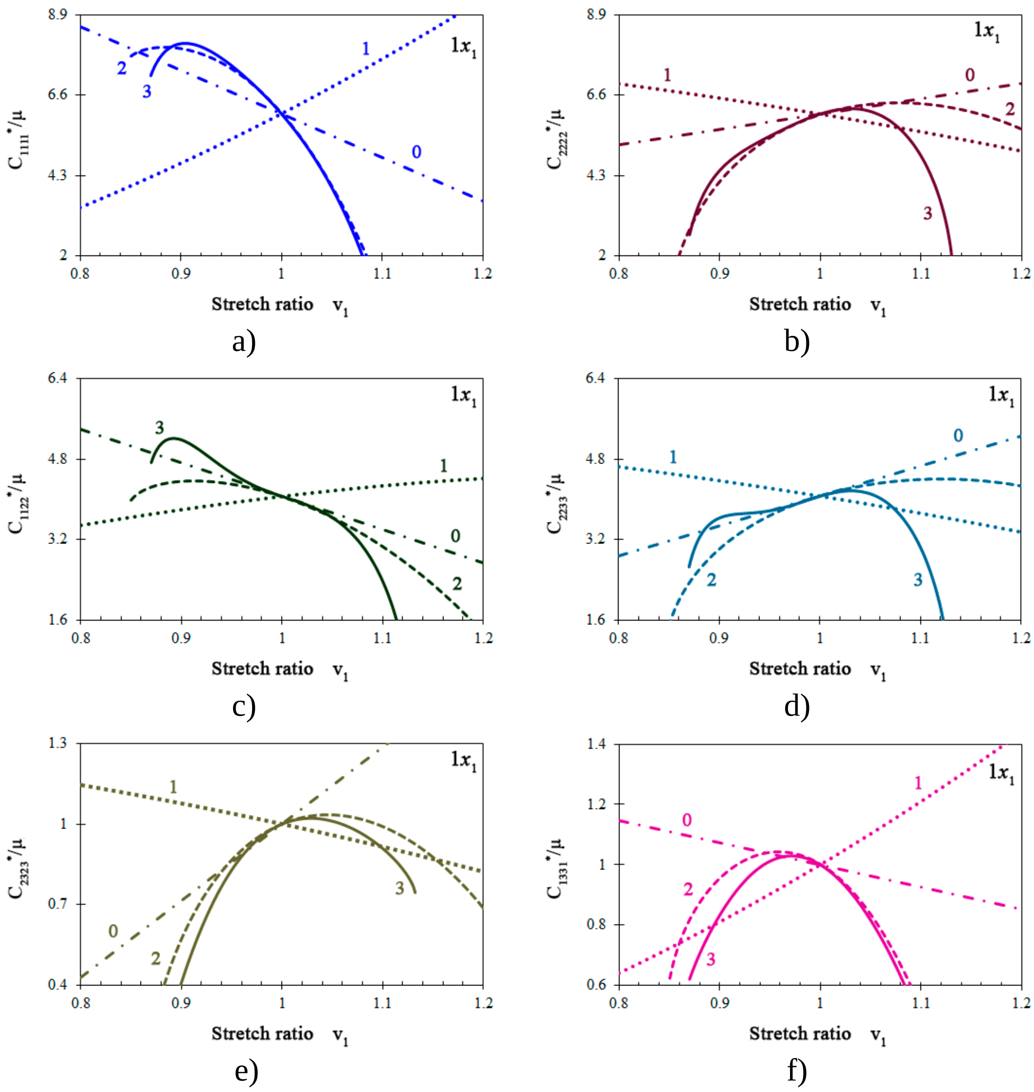

Fig. 5. The effect of accounting for nonlinearity of deformation on the coefficients C^*_{ijkl} in the case of uniaxial initial loading

It follows from the Figs. that the sensitivity of the various coefficients C^*_{ijkl} to the nonlinearity of the initial deformation is significantly different, but for most of the coefficients there exists a region of small deformations in which the nonlinearity can be neglected.

7. Conclusions

When assessing the dynamic, operational and strength characteristics of details made of artificial science-intensive and high-tech materials and working under the constant action of various kinds of loads, and when solving a wide range of applied problems, as well as problems of contact interaction, it is necessary to combine the most accurate accounting of material properties with the possibility of obtaining relatively simple and efficient solutions.

In this paper, within the theory of small deformations superposed on a finite one, a consistent linearization for the nonlinear equations of the mechanics of an elastic solid is carried out in the rectangular Lagrange coordinate system. The linearization is performed in a neighborhood of some initial stress state. We used the representation of the specific strain energy in terms of the algebraic invariants of the Green-Lagrange strain tensor, which take into account the elastic moduli of III and IV orders. Sufficiently simple and convenient expressions for the linearized equations of motion and the constitutive relations of the prestressed medium are derived, which allow one to take into account the nonlinear effects of the initial deformation on the elastic properties of the original material.

On the basis of the linearized constitutive relations obtained, the effect of accounting for nonlinearity of the initial deformation on the behavior of elastic coefficients of the originally isotropic material is studied for various types of the initial loading. The considerable difference in the behavior of the coefficients C^*_{ijkl} for two-, five- and nine-constant material models (Curves 1, 2, 3 and 1', 2', 3' in Figs. 3 and 4) is clearly shown. With simple initial loadings, for the majority of elastic coefficients there is a region of small deformations in which the differences between linear in deformation and nonlinear approximations, which take into account higher-order moduli (Curves 0, 2, 3 in Fig. 5; Curves 2, 3 and 2', 3' in Figs. 3 and 4), are insignificant. In the case of more complex loadings (Figs. 3b,d and 4b,d), the difference in the behavior of the coefficients C^*_{ijkl} becomes significant. Additionally, it has been shown that, in order to study the effect of initial loading on the elastic properties of an originally isotropic material, even at small deformations, the use of the linear approximation (39) is more preferable than the two-constant material model (Fig. 5).

It should be noted that the appropriateness of accounting for nonlinearity in the representation (37), (38) of the coefficients C^*_{ijkl} depends not only on the type of the stress state, the values of the initial stresses and the material used, but is largely determined by the problem posed and the characteristics studied (for example, characteristics of the wave field, parameters of the stress-strain state, characteristics of the dynamic processes).

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