Materials Physics and Mechanics 40 (2018) 254-260 Received: April 16, 2018

INDENTATION OF AN ELASTIC HALF-SPACE REINFORCED WITH A FUNCTIONALLY GRADED INTERLAYER BY A CONICAL PUNCH

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Abstract. An elastic half-space with a two-layered coating is considered. The upper layer is homogeneous while the lower layer is assumed to be made of a functionally graded material. Elastic moduli of the interlayer vary with depth according to arbitrary differentiable functions. The half-space is indented by a rigid conical punch. Approximated analytical expressions for the contact stresses are obtained using the bilateral asymptotic method. Expressions for the subsurface stresses and displacements are obtained in the form of some quadratures. Numerical results illustrating difference between the stress distributions for one- and two-layered coatings are presented.

Keywords: contact, indentation, conical punch, two-layered coating, functionally graded interlayer, elasticity, analytical methods

1. Introduction

The paper continues the study of contact mechanics of coatings reinforced with a functionally graded (FG) interlayer which was started by the authors in papers [1,2]. Such a structure of coatings may occur as the result of oxidation of a FG coating or may be created to obtain certain properties. For example, a coating consisting of an antifriction polymer composite attached to the load-bearing skeleton made of metal with complex nonmonotonic variation of elastic moduli is used to increase operating time and reduce wear of rails and wheel sets [3]. Most of the papers in the field of contact mechanics for FG materials address the case then the whole coating is made of a FG material [4–7]. Guler and Erdogan [4] analysed normal contact of a rigid punch and an elastic half-plane with FG coating with exponential variation of shear modulus. Ke, Wang, Liu and Zhang used piecewise linear approximation of the shear modulus to study 2D and axisymmetric contact of elastic solid with FG coating with arbitrary variation of elastic moduli. They also used similar approach to consider contact problems in more complicated formulations, for instance, thermoelastic frictional contact is considered in [7]. Coatings with FG interlayers have received less attention in research. Liu et al. [8] and Vasiliev et al. [2] considered torsion of an elastic half-space with a coating reinforced with a FG interlayer. Indentation of such a solid by a rigid flat-ended circular punch [1] and spherical punch [9,10] was also studied earlier. Guler et al. [11] considered a thin film bonded to a FG coating with exponentially varying elastic moduli on an elastic substrate.

http://dx.doi.org/10.18720/MPM.4022018_14

It has to be mentioned that methods for solution of contact problems for solids with FG coatings or piecewise homogeneous coatings or elastic layer on a rigid foundation are pretty similar. A regular asymptotic method was successfully used for thick coatings by Vorovich and Ustinov [12] and Zelentsov [13]. Methods based on the Wiener–Hopf factorization were effectively used for coatings (or layers) of small thickness, for instance, in [13,14]. The orthogonal polynomial method [15] and collocation method [16] are traditionally used to solve contact problems for intermediate thicknesses of the coating (layer). The generalized image method was developed and used by Fabrikant to solve several contact problems in [17–19].

The present paper addresses indentation by a rigid conical punch. The major difference from the results obtained in [4–11] is that the solution of integral equation of the problem is constructed in an approximated analytical form using the bilateral asymptotical method [20,21] while in [4–11] it is obtained numerically using collocation technique. The solution obtained in the paper is asymptotically exact for small and large values of relative coating thickness.

2. Statement of the problem

Let us consider an elastic half-space with a two-layered coating. The upper layer is homogeneous and has thickness h_1 while the lower layer (interlayer of the media) has thickness h_2 and made of a functionally graded material. Let us use a cylindrical coordinate system r, φ, z where z axis is normal to the coating surface and passes through the center of the punch. Lamé parameters of the half-space vary according to the following:

$$\{\mathrm{M},\Lambda\}(z)=\begin{cases}\{\mathrm{M}_0,\Lambda_0\}=\mathrm{const}, -h_1<z\le 0,\\ \{\mathrm{M}_1,\Lambda_1\}(z), \quad -H<z\le -h_1,\\ \{\mathrm{M}_2,\Lambda_2\}=\mathrm{const}, -\infty<z\le -H,\end{cases} \tag{1}$$

where $H=h_1+h_2$ is thickness of the coating, $M_1(z)$, $\Lambda_1(z)$ are arbitrary positive continuously differentiable functions. Here and after, superscripts 0, 1 and 2 correspond to the upper layer, to the interlayer and to the substrate, respectively.

A rigid conical punch is indented in the surface of the coated half-space by a normal centrally applied force P that causes elastic deformation of the half-space. Outside the contact area the surface is stress-free. The scheme of the contact is presented in the Fig. 1.

Let us introduce following notations: δ is the displacement of the punch, a is the radius of the contact area, α is the half slope of the cone, χ is the contact depth which satisfy following relation: $\cot\alpha=\chi/a$. The coating and the substrate are assumed to be glued without sliding. Therefore, the boundary conditions are:

$$z=0:\ \tau_{zr}^0=0,\ \sigma_z^0=0\ (r>a), w^0=-\delta+r\chi a^{-1}\ (r\le a). \tag{2}$$

$$z=-h_1:\tau_{zr}^0=\tau_{zr}^1,\ \sigma_z^0=\sigma_z^1,\ w^0=w^1,\ u^0=u^1, \tag{3}$$

$$z=-H:\tau_{zr}^1=\tau_{zr}^2,\ \sigma_z^1=\sigma_z^2,\ w^1=w^2,\ u^1=u^2. \tag{4}$$

The function to be determined is the contact normal pressure under the punch. To determine the radius of the contact region it is necessary to use an additional condition following from the continuity of the contact stress at the contact boundary:

$$\sigma_z\big|_{z=0}=-p_a(r),\ r\le a,\ p_a(a)=0. \tag{5}$$

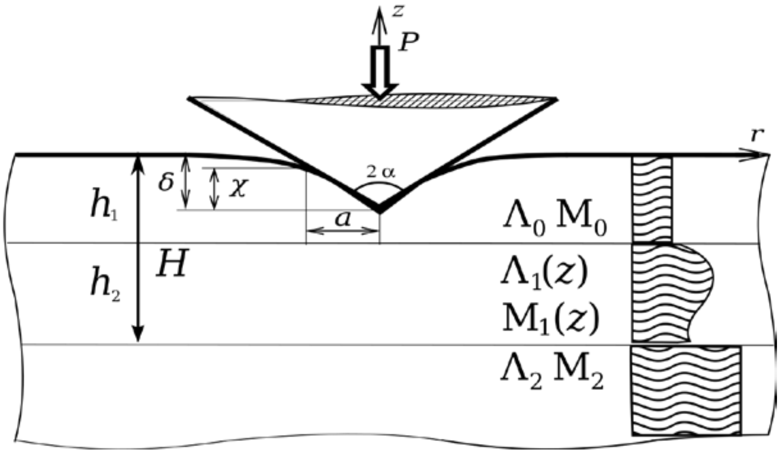


Fig. 1. Statement of the contact problem for an arbitrary functionally graded interlayer

The linear constitutive equations for an isotropic material are:

$$\sigma_r = 2\mathrm{M}(z)\frac{\partial u}{\partial r} + \Lambda(z)\theta, \sigma_\phi = 2\mathrm{M}(z)\frac{u}{r} + \Lambda(z)\theta, \sigma_z = 2\mathrm{M}(z)\frac{\partial w}{\partial z} + \Lambda(z)\theta,$$
$$\tau_{rz} = \mathrm{M}(z)\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right), \theta = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z}. \tag{6}$$

3. Solution of the problem

The problem is reduced to the solution of a following dual integral equation:

$$\begin{cases} \int_0^\infty P^*(\gamma)L(\lambda\gamma)J_0(r_0\gamma)d\gamma = \beta^{-1}\left(\delta\chi^{-1} - r_0\right), \ r_0 \le 1, \\ \int_0^\infty P^*(\gamma)J_0(r_0\gamma)\gamma d\gamma = 0, \ r_0 > 1. \end{cases} \tag{7}$$

This process was described in details in [26]. The following dimensionless variables were used above:

$$\{\lambda, r_0\} = \frac{\{H, r\}}{a}, \ z_0 = \frac{z}{H}, \ \beta = \frac{E_{ef}^{(2)}}{E_{ef}^{(0)}}, E_{ef}^{(0)} = \frac{E_0}{(1-\nu_0^2)}, E_{ef}^{(2)} = \frac{E_2}{(1-\nu_2^2)}, \{\delta^*, u^*, w^*\} = \frac{\{\delta, u, w\}}{\chi}$$
$$\{\Lambda^*, \mathrm{M}^*\}(z_0) = \frac{\{\Lambda, \mathrm{M}\}(Hz_0)}{E_{ef}^{(2)}}, \{\sigma_z^*, \sigma_r^*, \sigma_\varphi^*, \tau_{rz}^*, p^*\}(r_0) = \frac{2a\{\sigma_z, \sigma_r, \sigma_\varphi, \tau_{rz}, p_a\}(ar_0)}{E_{ef}^{(2)}\chi}. \tag{8}$$

$L(u)$ is the compliance function which is calculated numerically from a two-point boundary value problem for a system of ordinary differential equations with variable coefficients [1], E and v are Young's modulus and Poisson's ratio, $P^*(\gamma)$ is the Hankel transform of the dimensionless contact pressure. The solution of the integral equation (7) was constructed earlier [22] by using the bilateral asymptotic method [20]. For that purpose, the following approximation for the compliance function was used:

$$L(u) \approx L_N(u) = \prod_{i=1}^{N}\left(u^2 + A_i^2\right)/\left(u^2 + B_i^2\right). \tag{9}$$

Therefore, the solution of the integral equation has the form:

$$P^*(\gamma) = \sum_{i=1}^{N}\left[\frac{\gamma D_i + F(C_i, D_i, A_i\lambda^{-1})A_i\lambda^{-1}\sin\gamma - \gamma F(D_i, C_i, A_i\lambda^{-1})\cos\gamma}{\gamma\left(\gamma^2 + A_i^2\lambda^{-2}\right)}\right] + \frac{1-\cos\gamma}{\gamma^2}. \tag{10}$$

The contact stresses on the surface are determined by the formula:

$$p^*(r_0) = \ln\frac{1+\sqrt{1-r_0^2}}{r_0} + \sum_{i=1}^{N}\left(C_i \int_{r_0}^{1} \frac{\sinh(A_i\lambda^{-1}t)}{\sqrt{t^2-r_0^2}}dt + D_i \int_{r_0}^{1} \frac{\cosh(A_i\lambda^{-1}t)}{\sqrt{t^2-r_0^2}}dt \right), \tag{11}$$

where $F(x,y,z) = x\cosh(z) + y\sinh(z)$. Constants C_i and D_i (i=1,…, N) are the solution of the following system of linear algebraic equations:

$$\sum_{i=1}^{N} \frac{D_i}{A_i^2 - B_k^2} = \frac{1}{B_k^2}, \quad \sum_{i=1}^{N} C_i \frac{F(A_i, B_k, A_i\lambda^{-1})}{A_i^2 - B_k^2} = \frac{1}{B_k} - \sum_{i=1}^{N} D_i \frac{F(B_k, A_i, A_i\lambda^{-1})}{A_i^2 - B_k^2}, k = 1,.., N.$$

Displacements of the punch are obtained as a function of the relative coating thickness λ in the following form:

$$\delta^* = \frac{\pi}{2}\left(1 + \lambda \sum_{i=1}^{N} F\left(\frac{C_i}{A_i}, \frac{D_i}{A_i}, \frac{A_i}{\lambda}\right)\right) \tag{12}$$

Using (8) and (6) the following expressions for the displacements and stresses at internal points of the coated half-space are obtained:

$$u^* = \beta I_1,\ w^* = -\beta I_3,\ \sigma_z^* = \beta\left(\Lambda^*(z_0) I_6 - (\Lambda^*(z_0) + 2\mathrm{M}^*(z_0))\frac{I_4}{\lambda}\right),$$

$$\sigma_\varphi^* = \beta\left(2\mathrm{M}^*(z_0)\frac{I_1}{r_0} + \Lambda^*(z_0)\left(I_6 - \frac{I_4}{\lambda}\right)\right),\ \tau_{rz}^* = \beta \mathrm{M}^*(z_0)\left(\frac{I_2}{\lambda} + I_5\right), \tag{13}$$

$$\sigma_r^* = \beta\left((\Lambda^*(z_0) + 2\mathrm{M}^*(z_0)) I_6 - \Lambda^*(z_0)\frac{I_4}{\lambda} - 2\mathrm{M}^*(z_0)\frac{I_1}{r_0}\right).$$

The following notations for the quadratures were used above:

$$I_1(r_0,z_0) = \int_0^\infty P^*(\gamma) U^*(\lambda\gamma, z_0) \mathrm{J}_1(r_0\gamma) d\gamma,\ I_2(r_0,z_0) = \int_0^\infty P^*(\gamma) U'^*(\lambda\gamma, z_0) \mathrm{J}_1(r_0\gamma) d\gamma,$$

$$I_3(r_0,z_0) = \int_0^\infty P^*(\gamma) W^*(\lambda\gamma, z_0) \mathrm{J}_0(r_0\gamma) d\gamma,\ I_5(r_0,z_0) = \int_0^\infty P^*(\gamma) W^*(\lambda\gamma, z_0) \mathrm{J}_1(r_0\gamma)\gamma d\gamma, \tag{14}$$

$$I_4(r_0,z_0) = \int_0^\infty P^*(\gamma) W'^*(\lambda\gamma, z_0) \mathrm{J}_0(r_0\gamma) d\gamma,\ I_6(r_0,z_0) = \int_0^\infty P^*(\gamma) U^*(\lambda\gamma, z_0) \mathrm{J}_0(r_0\gamma)\gamma d\gamma.$$

Here $U^*(\gamma,z_0)$,$W^*(\gamma,z_0)$ and their derivatives with respect to z_0 are calculated numerically from the similar two-point boundary value problem for a system of ordinary differential equations with variable coefficients as the compliance function [1].

4. Numerical results

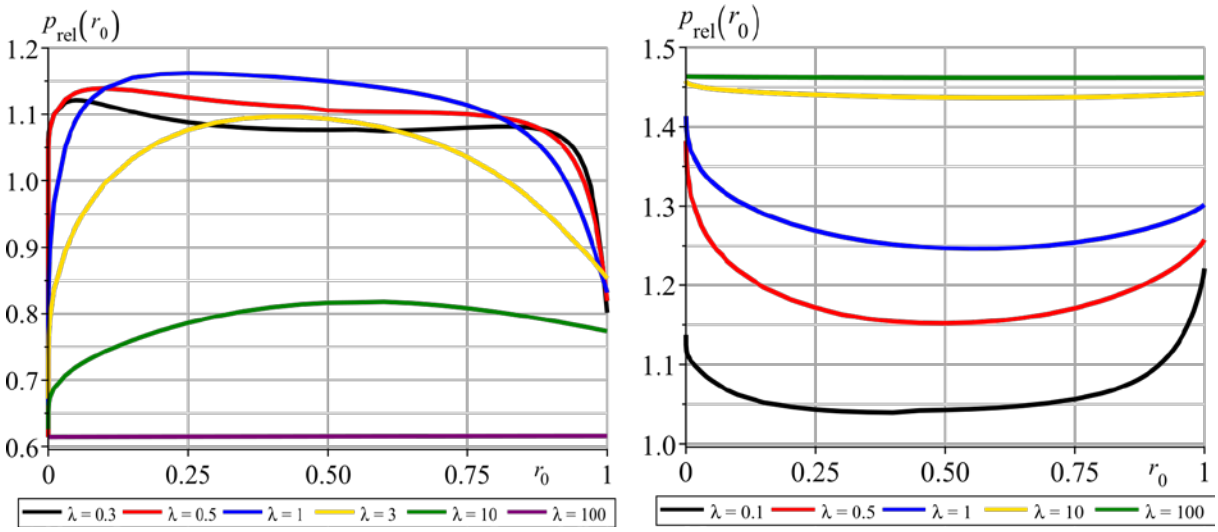
Let us consider the silicon substrate with the Young's modulus E = 146 GPa and the Poisson's ratio ν=0.22 and let the coating properties vary linearly from the pure nickel (E=203 GPa, ν = 0.31) on the surface to the pure silicon at depth. In addition, let us also consider the case when the coating was oxidized near its surface. As a result of oxidation a thin layer of NiO is occurred (E = 90 GPa, ν = 0.21). Some details of the creation and research of such a coating-substrate system, as well as the values of the elastic moduli, can be found in [23].

As it has been obtained earlier [1], the presence of the oxide upper layer sufficiently changes the compliance function. In particular, it has been shown that if the thickness of the oxide layer is small (h_1<0.2H) then the compliance function has nonmonotonic variation.

Figures 2*a* and 2*b* contain graphs of the distribution of relative contact stresses

$$p_{\mathrm{rel}}(r_0) = p^*(r_0) \Big/ \ln\frac{1+\sqrt{1-r_0^2}}{r_0}$$

for Ni/Si and NiO/Ni/Si in assumption that the Ni layer is oxidized to 5% of its thickness, i.e. $h_1 = 0.05H$. The relative contact stresses are more convenient for analysis because they have no singularities at r_0=0 and r_0=1. To calculate numerical results, the approximations of compliance functions by the expression (9) were constructed with the relative error less than 0.15%. That allows us to be confident in the high precision of the obtained numerical results.



a ***b***

Fig. 2. Distribution of the relative contact pressure for Ni/Si (*a*) and NiO/Ni/Si (*b*)

An important difference between the Ni and NiO/Ni coatings lays in the value of the "softness" parameter β. It is β=0.683<1 for Ni and β=1.63>1 for NiO/Ni. This parameter largely influences the distribution of the contact stresses in the vicinity of r_0=0 and r_0=1 especially for thin coatings [24]. As it can be seen from Fig. 2, the contact stresses near the points r_0=0 and r_0=1 decrease for NiO/Ni coating and increase for Ni coating, in comparison with the non-coated half-space. As λ→0 the contact stresses at any fixed value of r_0 tend to unit monotonically, for Ni coating, or nonmonotonically, for NiO/Si coating. This fact is the consequence of the similar behaviour of the corresponding compliance functions.

5. Conclusion

The indentation of an elastic half-space with a coating reinforced with a functionally graded interlayer by a rigid conical punch was studied. The distribution of the contact stresses for Ni/Si and NiO/Ni/Si was illustrated. It was shown that the presence of a thin oxide layer sufficiently changes the distribution of the contact stresses, especially near the central and boundary points of the contact region. The results of the paper can be easily generalized to the case of piezoelectric materials using recent results [25,26].

Acknowledgements. *The work is financially supported by the Federal Targeted Program for Research and Development in Priority Areas of Development of the Russian Scientific and Technological Complex for 2014-2020 under the contract No. 14.578.21.0246 (unique identifier RFMEFI57817X0246).*

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