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MODELING OF DENSELY CRACKED SURFACES AND THE GRIFFITH ENERGY CRITERION OF FRACTURE

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Abstract. Considering an elastic homogeneous isotropic body with a periodic family of surface microcracks, it is observed and justified rigorously that an influence of the microcracks on the far-field stress-strain state of the body can be taken into account at an appropriate asymptotic precision in a certain norm by creation of an asymptotic-variational model for an elastic dummy obtained by clipping out a thin near-surface layer of the elastic material. In other words, an abatement of a solid resistance due to the surface damage is equivalent to spalling of a subsurface flake realized in the model as a regular shift of the exterior boundary along the interior normal. The asymptotic-variational model is consistent with both, the Griffith energy criterion of fracture and spectral characteristics (e.g., eigenfrequencies) of the damaged body. At the same time, the traditional modelling through so-called "wall-laws" or singularly perturbed boundary conditions of Wentzel's type leads to ill-posed spectral problems. Numerical schemes for the asymptotic-variational model in the designed regularly perturbed domain do not differ from the ones in the original elastic body with a smooth intact surface that is without microcracks that makes the proposed approach to interpret damaged surfaces efficient.

1. Problem setting

Let a planar elastic body Ω occupy a domain in the plane $\mathbb{R}^2$ enveloped by a smooth simple closed contour $\Sigma = \partial\Omega$. In a neighborhood V of the surface Σ including its d-neighborhood V_d, $d > 0$, we introduce a system of the natural curvilinear coordinates (n, s) where n is the oriented distance to Σ, $n < 0$ in Ω, and s is the arc length along Σ. By rescaling we reduce to 1 the total length of the contour Σ so that all coordinates and geometric parameters become dimensionless.

Let N be a big natural number, hence $\delta = 1/N$ is a small parameter. Dealing with a smooth profile function H on Σ, we define a family of surface microcracks

$$\gamma_m^\delta(H) = \{x = (x_1, x_2) \in V: s = \delta m, n \in [-\delta H(s), 0]\}, \qquad m = 0, \dots, N-1, \tag{1}$$

and a two-dimensional elastic body with the densely cracked surface

$$\Omega^\delta(H) = \Omega \setminus \cup_{m=0}^{N-1} \gamma_m^\delta(H). \tag{2}$$

According to definition (1), the exterior boundary Σ_N^0 of the domain (2) consists of N open arcs of the small length $\delta = 1/N$ while the origin $s = 0$, cf. (1) at $m = 0$, could be fixed rather arbitrarily. Surfaces of the crack $\gamma_m^\delta(H)$ are denoted by $\gamma_m^{\delta\pm}(H)$ so that

while the boundary conditions (5) and (6) turn into

$$\sigma_{nn}\big(u^\delta;0,s\big)=g_n(s),\quad \sigma_{sn}\big(u^\delta;0,s\big)=g_s(s),\qquad s\in\Sigma_N^0, \tag{12}$$

$$\sigma_{sn}\big(u^\delta;n,m\delta\pm 0\big)=0,\ \ \sigma_{ss}\big(u^\delta;n,m\delta\pm 0\big)=0,\ \ n\in(-H(m\delta),0),\ \ m=0,\dots,N-1. \tag{13}$$

We will proceed in Section 2 with describing two-term asymptotics of displacements and stresses in $\Omega^\delta(H)$ which involve boundary layer terms appearing as decaying solutions of the elasticity problem in the cracked semi-strip $\Pi(H)=\Pi\backslash\Gamma(H)$,

$$\Pi=\left\{\xi=(\xi_1,\xi_2)\in\mathbb{R}^2:\xi_1<0,|\xi_2|<\tfrac{1}{2}\right\},\quad \Gamma(H)=\{\xi:\xi_1\in[-H,0],\xi_2=0\}. \tag{14}$$

Operating with this asymptotic form we will propose and investigate some models of the damage surface (3).

2. Asymptotics of elastic fields

Using an asymptotic procedure developed in [1, 2], we search for a solution of problem (4)-(6) in the asymptotic form

$$u^\delta(x)=u^0(x)+\delta\left(u'(x)+\chi(x)w(\delta^{-1}n,\delta^{-1}s;s)\right)+\tilde{u}^\delta(x), \tag{15}$$

where the main term u^0 is a solution of the limit ($\delta=0$) problem

$$-\partial_1\sigma_{1k}(u^0;x)-\partial_2\sigma_{2k}(u^0;x)=f_k(x),\qquad x\in\Omega,\quad k=1,2, \tag{16}$$

$$\sigma_{nn}(u^0;0,s)=g_n(s),\quad \sigma_{sn}(u^0;0,s)=g_s(s),\qquad s\in\Sigma. \tag{17}$$

Owing to the orthogonality conditions (9), this problem has a solution u^0 in the Sobolev space $H^2(\Omega)^2$ (the latter superscript 2 indicates the number of components in vector functions).

Let us describe correction terms in (15). The boundary layer term w is written in the stretched curvilinear coordinates

$$\xi=(\xi_1,\xi_2),\qquad\qquad \xi_1=\delta^{-1}n,\ \ \xi_2=\delta^{-1}s, \tag{18}$$

and is localized in the vicinity of Σ by means of a smooth cut-off function χ such that $\chi(x)=1$ for $|n|<d/3$ and $\chi(x)=0$ for $|n|>2d/3$. Furthermore, to satisfy the boundary conditions (6) on the crack surfaces, we impose the representation

$$w(\xi;s)=-\sigma_{ss}(u^0;0,s)W(\xi;H(s)), \tag{19}$$

where $W(\cdot\,;H(s))$ is a solution of the following problem in the elastic semi-strip Π with the surface crack $\Gamma(H)$ of length $H>0$, see (14),

$$-\partial_1\sigma_{1k}(W;\xi)-\partial_2\sigma_{2k}(W;\xi)=0,\qquad \xi\in\Pi(H)=\Pi\backslash\Gamma(H), \tag{20}$$

$$\sigma_{11}(W;0,\xi_2)=0,\ \ \sigma_{21}(W;0,\xi_2)=0,\qquad |\xi_2|<1/2, \tag{21}$$

$$\sigma_{22}(W;\xi_1,\pm 0)=\mp 1,\ \ \sigma_{12}(W;\xi_1,\pm 0)=0,\qquad \xi_1\in(-H,0), \tag{22}$$

$$W_k\left(\xi_1,+\tfrac{1}{2}\right)=W_k\left(\xi_1,-\tfrac{1}{2}\right),\ \sigma_{1k}\left(W;\xi_1,+\tfrac{1}{2}\right)=\sigma_{1k}\left(W;\xi_1,-\tfrac{1}{2}\right),\ k=1,2,\ \xi_1<0. \tag{23}$$

After the strong dilation, the coordinates (18) here are assumed to compose a Cartesian system so that $\partial_k=\partial/\partial\xi_k$ in (20) and the stresses $\sigma_{11}(W)=\delta\sigma_{nn}(W)$, $\sigma_{22}(W)=\delta\sigma_{ss}(W)$, $\sigma_{12}(W)=\delta\sigma_{ns}(W)$ are calculated according to formulas (7), instead of (10), with the corresponding replacement $x\longmapsto\xi$; notice that the factor δ results from the present usage of the stretched coordinates (18). The very reason for such the interpretation of coordinates (18) is based on the following observations. First, the Jacobian in V_d becomes $J(n,s)=1+\delta\xi_1\kappa(s)=1+O(\delta)$ and, therefore, we can omit the coefficient J^{-1} in all formulas (10) and

$$\left\|\nabla \tilde{u}^\delta;\ L^2\big(\Omega^\delta(H)\big)\right\| + \left\| \tilde{u}^\delta;\ L^2\big(\Omega^\delta(H)\big)\right\| \le c\delta^{3/2}. \qquad (30)$$

Here, $L^2\big(\Omega^\delta(H)\big)^2$ is the Lebesgue space so that the left-hand side of (30) exhibits the standard Sobolev norm in the space $H^1\big(\Omega^\delta(H)\big)^2$.

3. Modeling a damaged surface

We, first of all, observe that, owing to the exponential decay of the vector function W in (19), the Lebesgue norm of the boundary layer term $\delta\chi w$ is $O(\ \delta^{3/2})$ while the additional exponent ½ of δ comes in from the integral $\left(\int_{-d}^{0} e^{\tau n/\delta} dn\right)^{1/2} = O(\ \delta^{1/2})$ which appears with some $\tau > 0$ just due to the above-mentioned decay property. Thus, the estimate (30) without this term can be converted into

$$\left\| u^\delta - v^\delta\ ;\ L^2\big(\Omega^\delta(H)\big)\right\| \le c\delta^{3/2}, \qquad (31)$$

where the sum v^δ of two regular terms from (15) figure in, namely

$$v^\delta(x) = u^0(x) + \delta u'(x). \qquad (32)$$

Notice that in view of the stretched coordinates (15) in the boundary layer term $w(\xi; s)$, absent in (32), the norm of gradient of the difference $u^\delta - v^\delta$ has order $\delta^{1/2}$ and therefore the Sobolev norm cannot substitute for the Lebesgue one in (31). In other words, the simplistic estimate (31) is much weaker than the original estimate (30). However, getting rid of the boundary layer term in the sum (32) makes the asymptotic approximation (32) much simpler in comparison with the full asymptotic expansion (15). This simplification can be used to create primary asymptotic-variational models which do not care about localized effects, i.e., the near-field, but about the far-field approximation only. Let us first discuss a traditional model involving boundary conditions of Wentzel's type, also known in the mathematical literature as "wall-laws".

We compose the boundary value problem in the intact that is without the mocrocracks (1), planar solid Ω

$$-\partial_1 \sigma_{1k}\big(\hat{v}^\delta; x\big) - \partial_2 \sigma_{2k}\big(\hat{v}^\delta; x\big) = f(x), \qquad x \in \Omega, \quad k = 1,2\,, \qquad (33)$$

$$\sigma_{nn}\big(\hat{v}^\delta; 0, s\big) = \hat{g}_n^\delta(s) + \delta\kappa(s)\hat{\alpha}(s)\varepsilon_{ss}\big(\hat{v}^\delta; 0, s\big),$$

$$\sigma_{sn}\big(\hat{v}^\delta; 0, s\big) = \hat{g}_s^\delta(s) - \delta\partial_s\left(\hat{\alpha}(s)\varepsilon_{ss}\big(\hat{v}^\delta; 0, s\big)\right), \quad s \in \Sigma, \qquad (34)$$

where

$$\hat{\alpha}(s) = M\alpha(s) = M^2 E_{H(s)} > 0 \qquad (35)$$

and

$$\hat{g}_n^\delta(s) = g_n(s) - \delta L\kappa(s)\hat{\alpha}(s)g_s(s),\ \hat{g}_s^\delta(s) = g_s(s) + \delta L\partial_s\big(\hat{\alpha}(s)g_s(s)\big). \qquad (36)$$

The boundary conditions (34) were constructed in two steps. First, we multiply (25) with δ and add the result to (17). Second, we use the notation (32) together with formula (28) and, furthermore, replace $v^\delta(x)$ and $\delta u^0(x)$ by $\hat{v}^\delta(x)$ and $\delta\hat{v}^\delta(x)$ respectively.

Since one of the boundary conditions (35) involves a second-order differential operator on the boundary with the small coefficient δ, the problem is to be regarded as a singularly perturbed problem of Wentzel's type. At the same time, the sum (32) leaves a small, of order δ^2, discrepancy and the asymptotic precision estimate

$$\left\|\nabla\big(\hat{v}^\delta - v^\delta\big)\ ;\ L^2(\Omega)\right\| + \left\|\hat{v}^\delta - v^\delta\ ;\ L^2(\Omega)\right\| \le c\delta^2 \qquad (37)$$

$$g^{\delta}_{s^{\delta}_{D}}(s) = g_s(s) + \delta\big(g'_s(s) - \partial_s D(s) g_n(s)\big) + O(\delta^2),$$

where $s^{\delta}_{D} = \Big(-\big(n^{\delta}_{D}\big)_{s}, \big(n^{\delta}_{D}\big)_{n}\Big)$ stands for the tangential unit vector at the contour Σ, cf., formulas (42) for the normal vector. Relation (44) inherited from (10) and (42), are inserted into the boundary conditions (41) and, after applying the Taylor formula in the variable n, we collect coefficients of δ and obtain

$$\sigma_{nn}(\breve{u}'; 0, s) = D(s)\partial_n \sigma_{nn}(u^0; 0, s) - 2\partial_s D(s)\sigma_{ns}(u^0; 0, s) + g'_n(s) + \partial_s D(s) g_s(s),$$

$$\sigma_{ns}(\breve{u}'; 0, s) = D(s)\partial_n \sigma_{ns}(u^0; 0, s) - 2\partial_s D(s)\big(\sigma_{nn}(u^0; 0, s) - \sigma_{ss}(u^0; 0, s)\big) +$$

$$+ g'_n(s) + \partial_s D(s) g_s(s). \quad (45)$$

Setting $n = 0$ in the differential equations (16) rewritten in the form (11) yields

$$\partial_n \sigma_{nn}(u^0; 0, s) = -f_n(0, s) - \partial_s \sigma_{ns}(u^0; 0, s) - \kappa(s)\big(\sigma_{nn}(u^0; 0, s) - \sigma_{ss}(u^0; 0, s)\big),$$

$$\partial_n \sigma_{ns}(u^0; 0, s) = -f_s(0, s) - \partial_s \sigma_{ss}(u^0; 0, s) - 2\kappa(s)\sigma_{ns}(u^0; 0, s). \quad (46)$$

Taking (46) into account, we fix

$$g'_n(s) = D(s)\big(f_n(0, s) + \partial_s g_s(s) + \kappa(s) g_n(s)\big) - \partial_s D(s) g_s(s),$$

$$g'_s(s) = D(s)\big(f_s(0, s) - 2\kappa(s) g_s(s)\big) - \partial_s D(s)\big(g_n(s) - g_s(s)\big), \quad (47)$$

and, therefore, the equations (45) become

$$\sigma_{nn}(\breve{u}'; 0, s) = -\kappa(s) D(s)\sigma_{ss}(u^0; 0, s), \sigma_{ns}(\breve{u}'; 0, s) = \partial_s\big(D(s)\,\sigma_{ss}(u^0; 0, s)\big),\ s \in \Sigma. \quad (48)$$

Remark. In the case $g = 0$ the necessity to introduce the addendum $\delta g'(s) = \delta D(s) f(0, s)$, see (47), onto the right-hand side of (44) is explained by the asymptotic representation

$$\int_{\Omega^{\delta}_{D}} f(x)dx = \int_{\Omega} f(x)dx - \int_{\Omega\setminus\Omega^{\delta}_{D}} f(x)dx = \int_{\Omega} f(x)dx - \delta \int_{\Sigma} D(s) f(0, s) ds +$$

$$+ O(\delta^2). \qquad \blacksquare$$

Comparing (25) and (48), we conclude that in the case

$$D(s) = \alpha(s), \quad s \in \Sigma, \quad (49)$$

the main correction terms $\breve{u}'$ in the ansatz (43) and u' in the ansatz (15) coincide with each other, that is $\breve{u}' = u'$. Moreover, the evident estimate of the remainder $\tilde{\breve{u}}^{\delta}(x)$ in (43)

$$\| \breve{u}^{\delta} - u^0 - \delta\, \breve{u}'\, ;\ L^2\big(\Omega^{\delta}_{D}\big)\| \leq c\delta^{3/2}, \quad (50)$$

cf., [4; Section 7.6.5], combined with (31) and (32), demonstrates that

$$\| u^{\delta} - \breve{u}^{\delta}\, ;\ L^2\big(\Omega^{\delta}_{D}\big)\| \leq c\delta^{3/2}. \quad (51)$$

In other words, the solution of the model problem (40), (41) in the domain Ω^{δ}_{D} with a regularly perturbed but smooth boundary Γ^{δ}_{D} gives a good approximation, although in the weaker norm of $L^2\big(\Omega^{\delta}_{D}\big)^2$, for the solution of the original problem (4)-(6) in the domain $\Omega^{\delta}(H)$ with singularly perturbed piecewise smooth boundary $\Gamma^{\delta}(H)$, namely the damaged surface (3). Beyond any doubt solving numerically a problem in a smooth domain is much simpler than in a domain with rapidly oscillating boundary and plenty of corner points. The parameter $D(s)$, that is depth of boundary's shift along the inward normal, in the regular perturbation (39) of the reference domain Ω is computed, according to (49), (27), (24), through the solution $W(\xi; H(s))$ of a standard elasticity boundary value problem in the cracked semi-strip (14) and the energy functional (24) can be tabulated. Moreover, in Section 6 it will be observed that the integral

eigenvalues give rise to pure imaginary eigenfrequencies that have no physical sense. The model (55), (56) in the regularly perturbed domain Ω_D^δ is deprived of this flaw. Moreover, numerical schemes to solve the variational spectral problem (58) in the smooth domain Ω_D^δ require for alike computer expenses as for the spectral elasticity problem in the original unperturbed body Ω that is without any damage while it is incomparably more complicated to solve the original spectral problem in the domain damaged with the family (1) of surface microcracks. It should be mentioned that each of eigenvalues in (54) can be regarded as a functional on the corresponding eigenmode. In this way an error estimate in the Lebesgue norm is sufficient to conclude with the inequality (59).

6. The Griffith energy criterion of fracture

First of all, we mention that the positive function $(0,+\infty) \ni H \to E_H$ from (24) is strictly monotone increasing. To make this conclusion, we consider the minimization of the potential energy functional for the problem (20)-(23)

$$\min\left\{E_H(U) - \sum_\pm \int_{-H}^0 U_2(\xi_1, \pm 0) d\,\xi_1 : \ \ U \in \mathcal{B}_H \coloneqq H^1(\Pi(H))^2 \right\} \tag{60}$$

and observe that the solution $W(\xi; H)$ of this problem is just a unique minimizer for (60) while, according to the Green formula in $\Pi(H)$, we have

$$E_H(W) - \sum_\pm \int_{-H}^0 W_2(\xi_1, \pm 0) d\,\xi_1 = -E_H(W) =: -E_H. \tag{61}$$

When the crack $\Gamma(H)$ grows, the function space where test vector functions are taken from, enlarges, that is $\mathcal{B}_{H+\Delta H} \supset \mathcal{B}_H$ for $\Delta H > 0$ because the extended crack allows for jumps of elastic fields between its surfaces. Thus, the change $H \mapsto H + \Delta H$ makes the minimum in (60) smaller but, in view of formula (61) with minus sign on the right, the desired relation $E_{H+\Delta H} > E_H$ holds true. It is worth here to mention a result in [8], namely application of the unit loading on surfaces of the incremental cut $\Gamma(H+\Delta H)\backslash\Gamma(H)$ of the small length ΔH inputs into the functional (60) an infinitesimal value $O\left(|\Delta H|^{3/2}\right)$ of a higher order.

Notice that the classical Griffith formula allows to express the energy increment $E_{H+\Delta H}(W) - E_H(W)$ through the stress intensity factor at the tip of the crack $\Gamma(H)$.

We assume that the profile function $H(s)$ in (1) gets a small positive increment $\Delta H(s)$ and therefore all microcracks have grown a bit. Putting $\mathcal{H}(s) = H(s) + \Delta H(s)$, we denote by $\mathcal{U}^\delta(x)$ the solution of the problem (4)-(6) in the perturbed domain $\Omega^\delta(\mathcal{H})$ and set

$$a(s) = ME_{\mathcal{H}(s)}. \tag{62}$$

Furthermore, $\breve{\mathcal{U}}^\delta(x)$ stands for the solution of the problem (40), (41) in the domain Ω_D^δ given by formula (39) with the new profile function $\mathcal{D}(s) = a(s)$, see (62) and (49).

The total increment $\Delta\mathcal{L}$ of the cracks length can be calculated as follows:

$$\Delta\mathcal{L} = \sum_{m=0}^{N-1} \delta\Delta H(m\delta) = \delta\int_\Sigma \Delta H(s)\, ds + O(\delta^2) \tag{63}$$

and thus, according to the Griffith formula, the increment $\Delta\mathcal{S}$ of the surface energy $\mathcal{S}$ becomes equal to

$$\Delta\mathcal{S} = 2s\delta\int_\Sigma \Delta H(s)\, ds + O(\delta^2). \tag{64}$$

Here, $s > 0$ is the density of the surface energy. Notice that, recalling the rescaling made in Section 1, the relation (63) can be rewritten in the form

$$\Delta\mathcal{S} = 2s\,\Delta\mathcal{V} + O(\delta^2) = \ v\,\Delta\mathcal{V} + O(\delta^2), \tag{65}$$

where $\Delta\mathcal{V} > 0$ is the decrement of the volume of the elastic dummy body (39) due to the growth of the microcracks. In other words, $\Delta\mathcal{V}$ is the difference of the areas of Ω_D^δ and $\Omega_\mathcal{D}^\delta$. At the same time, the factor $\mathscr{v} = 2\mathscr{s}$ on the right-hand side of (65) can be considered as the density of the volume energy caused by spilling elastic material off. In other words, $\mathscr{v}$ is the aggregate crack surface per volume unit.

To simplify intermediate calculations, we suppose for a while that the traction g on the exterior surface Σ_N^0 of the damaged body $\Omega^\delta(H)$ is absent and the volume forces vanish in the d-neighborhood V_d of Σ_N^0. Hence, the right-hand sides in the boundary conditions (5) and (41) vanish. Then, similarly to (61), the potential energies of the bodies $\Omega^\delta(\mathcal{H})$ and $\Omega_\mathcal{D}^\delta$ are calculated as follows:

$$\mathcal{P}^\delta(\mathcal{H}) = -\frac{1}{2}\sum_{k=1,2}\int_\Omega \mathcal{U}_k^\delta(x) f_k(x)dx,\ \mathcal{P}_\mathcal{D}^\delta = -\frac{1}{2}\sum_{k=1,2}\int_\Omega \breve{\mathcal{U}}_k^\delta(x) f_k(x)dx. \tag{66}$$

We emphasize that the integration domain Ω is the same in both the integrals because $f_k = 0$ in $\Omega^\delta(\mathcal{H}) \cap V_d$ as well as in $\Omega_\mathcal{D}^\delta \cap V_d$. Moreover, using the estimate (51) relating the displacement fields $\mathcal{U}^\delta(x)$ in the cracked body $\Omega^\delta(\mathcal{H})$ and $\breve{\mathcal{U}}^\delta(x)$ in the elastic dummy $\Omega_\mathcal{D}^\delta$ we derive that

$$\left|\mathcal{P}^\delta(\mathcal{H}) - \mathcal{P}_\mathcal{D}^\delta\right| \le c\delta^{\frac{3}{2}}. \tag{67}$$

This error estimate again gets a supplementary smallness order and remains valid even in the general case of non-nil traction g. The formula (67) applied for $\mathcal{H}(s) = H(s) + \Delta H(s)$ and $\mathcal{H}(s) = H(s)$ leads to the relationship

$$\left|\Delta\mathcal{P}^\delta(\mathcal{H}) - \Delta\mathcal{P}_\mathcal{D}^\delta\right| \le c\delta^{\frac{3}{2}} \tag{68}$$

between increments of potential energy in the damaged body $\Omega^\delta(\mathcal{H})$ and in the dummy $\Omega_\mathcal{D}^\delta$, that is

$$\Delta\mathcal{P}^\delta(H) = \mathcal{P}^\delta(H + \Delta H) - \mathcal{P}^\delta(H), \qquad \Delta\mathcal{P}_\mathcal{D}^\delta = \mathcal{P}_{\mathcal{D}+\Delta D}^\delta - \mathcal{P}_\mathcal{D}^\delta$$

with a clear meaning of the increment ΔD of the profile function in (39).

As a result, the inequality (68) together with relations (64) and (65) helps us to convert the Griffith formula of energy balance for propagating cracks

$$\Delta\mathcal{P}^\delta(H) + 2\mathscr{s}\Delta\mathcal{L} \approx 0$$

into the following formula of energy balance

$$\Delta\mathcal{P}_\mathcal{D}^\delta + \mathscr{v}\,\Delta\mathcal{V} \approx 0.$$

The latter demonstrates that the asymptotic-variational model (40), (41) in the domain (39) gets a good match with the Griffith criterion of fracture so that shrinkage of the elastic dummy $\Omega_\mathcal{D}^\delta$ due to the damage spread caused by growth of surface microcracks is governed by the Griffith energy criterion, too.

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