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A STATIC BOUNDARY ELEMENT ANALYSIS OF 3D ANISOTROPIC ELASTIC PROBLEMS

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Abstract. This paper presents a direct boundary element approach for anisotropic static three-dimensional linear elastic problems. Formulation is based on the use of regularized boundary integral equation (BIE) for displacements. This BIE is weakly singular which is advantageous compared to the traditional strongly singular formulations. The displacement static fundamental solution is expressed in terms of an integral over a circumference with a unit radius. For the efficient numerical implementation of these fundamental solutions an interpolation scheme is used. For the spatial discretization a mixed approximation of geometry and boundary fields is employed. Numerical solutions of the problem of spherical cavity in an infinite elastic medium subjected to the uniform internal pressure are presented for the materials with different degrees of anisotropy.

Keywords: anisotropic elasticity, static problems, boundary element method, Green's functions

1. Introduction

In many critical applications in various engineering fields like aerospace, acoustical, geotechnical and civil engineering the use of anisotropic elastic materials are essential for designing efficient structural elements. For the most three-dimensional practical problems numerical modeling is the only way to obtain reliable and accurate results.

Boundary-value problems of linear elasticity are a classical topic and they can be solved by a variety of numerical methods like finite element method, boundary element method (BEM) and finite difference method. The key to successful application of the conventional BEM is the availability of fundamental solutions (or Green's functions). By using the Radon transform Wang and Achenbach [1,2] obtained an integral representation of the static Green's functions in anisotropic media. Since numerical integration of the line integral appearing in such representation of the fundamental solutions must be done for each pair of the source and field points it is unreasonable to directly implement this kind of approach of calculating the anisotropic elastic static Green's functions into the BEM formulation. To speed up the boundary element procedure Wilson and Cruse [3] proposed a simple and efficient way: fundamental solutions and their spatial derivatives for considered anisotropic elastic materials are calculated in advance and stored in a database and then for the particular pair of the source and the field points the values of the fundamental solutions are obtained by the interpolation. Because of the mathematical complexity of Green's functions for anisotropic elasticity and issues associated with their numerical implementation, number of known of different BEM approaches for a static analysis of three-dimensional elastic problems [4-12] is far less than for the isotropic ones.

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In this work, a direct boundary element technique for solving three-dimensional anisotropic elastic problems is presented. Formulation is based on the use of regularized boundary integral equation (BIE) for displacements. This BIE is weakly singular which is advantageous compared to the traditional formulations in which an integration of the strongly singular integral is required. The displacement static fundamental solution is expressed in terms of an integral over a circumference with a unit radius. For the efficient numerical implementation of these fundamental solutions an interpolation scheme is employed. For the spatial discretization a mixed approximation of geometry and boundary fields is employed. Numerical solutions of the problem of spherical cavity in an infinite elastic medium subjected to the uniform internal pressure are presented for the materials with different degrees of anisotropy.

2. The boundary element formulation for 3D anisotropic elastostatics

Consider a homogeneous three-dimensional anisotropic linear elastic solid $\Omega \subset R^3$ with surface $S = \partial\Omega$. Neglecting body forces, the static equilibrium is

$$\sigma_{ij,j}(\mathbf{x}) = 0, \ \ \mathbf{x} \in \Omega, \ \ i, j = \overline{1,3}, \tag{1}$$

where σ is the Cauchy stress tensor.

The stress and strain are related through the constitutive equation

$$\sigma_{ij}(\mathbf{x}) = C_{ijkl}\varepsilon_{kl}(\mathbf{x}), \ \ k, l = \overline{1,3}, \tag{2}$$

where C_{ijkl} is the fourth order anisotropic material elasticity tensor.

The linear geometry relation is defined by:

$$\varepsilon_{kl}(\mathbf{x}) = \frac{1}{2}\left(u_{k,l}(\mathbf{x}) + u_{l,k}(\mathbf{x})\right), \tag{3}$$

where ε_{kl} is the strain tensor and u_k is the displacement vector.

For the outward unit normal vector n_j the traction vector t_i is defined by

$$t_i(\mathbf{x}) = \sigma_{ij}(\mathbf{x}) n_j(\mathbf{x}). \tag{4}$$

By combining equations (1)-(3) the field equations of elastostatics are obtained

$$C_{ijkl} u_{k,il}(\mathbf{x}) = 0, \ \ \mathbf{x} \in \Omega. \tag{5}$$

In order to formulate and solve a particular boundary-value problem, boundary conditions are needed

$$u_i(\mathbf{x}) = u_i^*(\mathbf{x}), \ \ \mathbf{x} \in S_u, \tag{6}$$

$$t_i(\mathbf{x}) = t_i^*(\mathbf{x}), \ \ \mathbf{x} \in S_t, \tag{7}$$

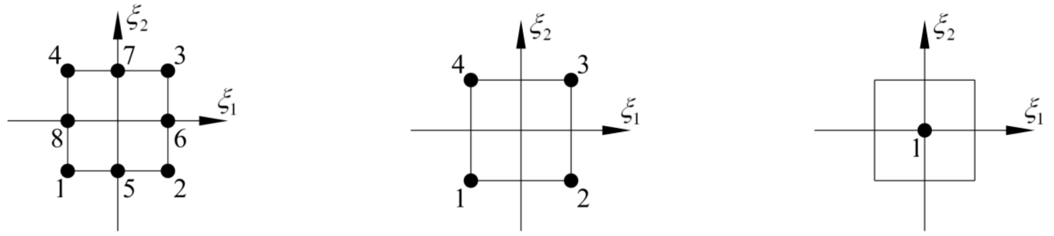
with $u_i^*(\mathbf{x})$ and $t_i^*(\mathbf{x})$ being the prescribed values of the displacements and tractions, respectively.

A weakly singular boundary integral equation equivalent to the Eq. (5) reads as follows

$$\int_\Gamma h_{jk}(\mathbf{r})\left[u_k(\mathbf{y}) - u_k(\mathbf{x})\right]\mathrm{d}S(\mathbf{y}) - \int_\Gamma t_k(\mathbf{y}) g_{jk}(\mathbf{r})\mathrm{d}S(\mathbf{y}) = 0, \tag{8}$$

with $\mathbf{r} = \mathbf{y} - \mathbf{x}$, $\mathbf{x}$ is the source point and $\mathbf{y}$ is the field point; g_{jk} and h_{jk} are the three-dimensional static displacement and traction anisotropic fundamental solutions, respectively.

The boundary element method is employed to numerically solve Eq. (8). The boundary surface S is discretized with quadrangular boundary elements. The geometry of each element is approximated by the quadratic shape functions. The boundary fields: displacements u_i and tractions t_i on boundary elements are interpolated by using different functions (linear for displacements and constant for tractions) and their corresponding nodal values. Locations of nodes for the approximation of geometry, displacements and tractions on the reference element with the local coordinates $-1 \le \xi_1, \xi_2 \le 1$ are depicted in Fig. 1. This is the simplest possible discretization scheme which allows splitting nodal points of displacements and tractions and thus maintaining continuity of displacements across adjacent boundary elements and at the same time to properly model discontinuous tractions, since the collocation points in this case lie in the centers of the elements.



a *b* *c*

Fig. 1. Locations of the nodes for approximation a) geometry, b) displacements, c) tractions

Then this so-called mixed discretization described in major details in [13,14] is applied to the boundary integral equation (8). By collocating discretized version of Eq. (8) at the nodes of the unknown boundary fields, applying the prescribed boundary conditions and rearranging the known and unknown quantities, a final resolving system of linear algebraic equations can be obtained:

$$\mathbf{Ap} = \mathbf{f}, \tag{9}$$

where $\mathbf{A} \in \mathrm{R}^{\mathrm{N_{dof}} \times \mathrm{N_{dof}}}$ is the rearranged dense and asymmetric system matrix, $\mathbf{f} \in \mathrm{R}^{1 \times \mathrm{N_{dof}}}$ consist of unknown boundary displacements and tractions known boundary data and elements of $\mathbf{p} \in \mathrm{R}^{\mathrm{N_{dof}} \times 1}$ are linear combination according to the prescribed boundary data of integrals of the kernels in Eq. (8); $\mathrm{N_{dof}}$ is the total number of unknowns.

3. Green's functions

An integral representation for the three-dimensional static anisotropic elastic displacement fundamental solution can be written as follows

$$g_{ij}(\mathbf{r}) = \frac{1}{8\pi^2 |\mathbf{r}|} \int_{|\mathbf{d}|=1} \Gamma_{ij}^{-1}(\mathbf{d}) dL(\mathbf{d}), \tag{10}$$

where $\Gamma_{ij}(\mathbf{d}) = C_{kijl} d_k d_l$ and the domain of the integration is the unit circumference defined by intersection of the unit sphere centered at the source point **x** and the plane perpendicular to **r** and passing through the point **x** (for detailed derivation see, e.g. [1,2]). For a set of orthonormal vectors **p** and **q** in this plane and angle $0 \le \varphi \le 2\pi$ we can express vector $\mathbf{d}(\varphi)$ (see Fig. 2) which defines the domain of integration $|\mathbf{d}| = 1$ as

$$\mathbf{d}(\varphi) = \cos(\varphi)\mathbf{p} + \sin(\varphi)\mathbf{q}, \ 0 \le \varphi \le 2\pi, \tag{11}$$

$$\mathbf{p} \cdot \mathbf{e} = 0, \ \mathbf{q} \cdot \mathbf{e} = 0, \ \mathbf{p} \cdot \mathbf{q} = 0, \ \mathbf{e} = \frac{\mathbf{r}}{|\mathbf{r}|}, \ \mathbf{e} = [e_1, e_2, e_3]. \tag{12}$$

We choose **p** and **q** as follows

$$\mathbf{p} = \frac{[e_2, -e_1, 0]}{\sqrt{1-e_3^2}}, \ \mathbf{q} = \mathbf{e} \times \mathbf{p} = \frac{[e_1 e_3, e_2 e_3, -(1-e_3^2)]}{\sqrt{1-e_3^2}}. \tag{13}$$

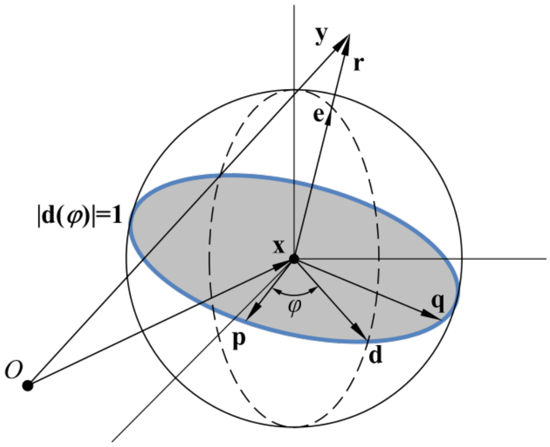


Fig. 2. Geometrical representation of domain of integration $\left|\mathbf{d}(\varphi)\right| = 1$

Substituting (13) into (11) we get

$$\mathbf{d}(\varphi) = \frac{\left[e_2\cos\varphi + e_1e_3\sin\varphi, -e_1\cos\varphi + e_2e_3\sin\varphi, -\left(1-e_3^2\right)\sin\varphi\right]}{\sqrt{1-e_3^2}}, \quad \mathbf{d}\cdot\mathbf{r} = 0. \qquad (14)$$

With the use of the generalized Hooke's law the traction fundamental solution are defined as

$$h_{jp}(\mathbf{r}) = C_{ijkl}\, g_{kp,l}(\mathbf{r}) n_i(\mathbf{y}), \quad p = \overline{1,3}, \quad \mathbf{y}\in S, \qquad (15)$$

where $n_i(\mathbf{y})$ outward unit normal vector to the boundary surface S.

Static anisotropic elastic displacement fundamental solution and its derivative can be expressed in spherical coordinates (r,θ_1,θ_2) as follows

$$g_{ij}(r,\theta_1,\theta_2) = \frac{H_{ij}(\theta_1,\theta_2)}{r}, \quad g_{ij,k}(r,\theta_1,\theta_2) = \frac{H_{ijk}(\theta_1,\theta_2)}{r^2}, \quad -\pi \le \theta_1 < \pi, \quad 0 \le \theta_2 \le \pi. \qquad (16)$$

Since the $H_{ij}(\theta_1,\theta_2)$ and $H_{ijk}(\theta_1,\theta_2)$ are functions of only two coordinates θ_1 and θ_2, their values at any $\left(\theta_1,\theta_2\right)$ can be obtained using any suitable interpolation scheme [3]. In this work, we employ linear Lagrange interpolation on a uniform rectangular grid.

4. Numerical example

Consider a spherical cavity with radius $R = 1$ m in an infinite elastic medium (see Fig. 3). The cavity is subjected to the uniformly distributed normal internal pressure $P = 10^7$ Pa. Displacements at the boundary surface are studied. Static anisotropic fundamental solutions are computed at the field points using the interpolation scheme on the grid of 800×800 nodes.

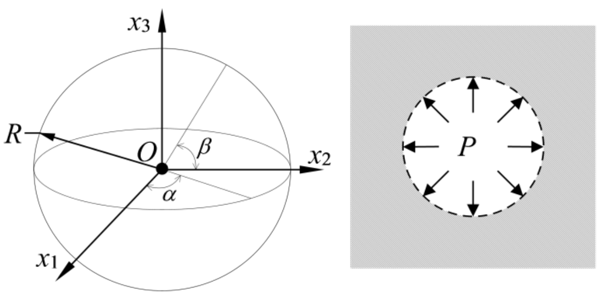


Fig. 3. Spherical cavity under internal pressure

Problem is considered for four different materials: (i) isotropic aluminum polycrystal, (ii) cubic aluminum crystal, (iii) orthorhombic barium sodium niobate and (iv) trigonal sapphire. The elasticity tensors for these materials are given as follows [10]

$$\mathbf{C}^{\mathrm{i}} = \begin{bmatrix} 111 & 61 & 61 & 0 & 0 & 0 \\ 61 & 111 & 61 & 0 & 0 & 0 \\ 61 & 61 & 111 & 0 & 0 & 0 \\ 0 & 0 & 0 & 25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 25 & 0 \\ 0 & 0 & 0 & 0 & 0 & 25 \end{bmatrix} \mathrm{GPa}, \tag{17}$$

$$\mathbf{C}^{\mathrm{ii}} = \begin{bmatrix} 108 & 61.3 & 61.3 & 0 & 0 & 0 \\ 61.3 & 108 & 61.3 & 0 & 0 & 0 \\ 61.3 & 61.3 & 108 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.85 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2.85 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2.85 \end{bmatrix} \mathrm{GPa}, \tag{18}$$

$$\mathbf{C}^{\mathrm{iii}} = \begin{bmatrix} 239 & 104 & 50 & 0 & 0 & 0 \\ 104 & 247 & 52 & 0 & 0 & 0 \\ 50 & 52 & 135 & 0 & 0 & 0 \\ 0 & 0 & 0 & 65 & 0 & 0 \\ 0 & 0 & 0 & 0 & 66 & 0 \\ 0 & 0 & 0 & 0 & 0 & 76 \end{bmatrix} \mathrm{GPa}, \tag{19}$$

$$\mathbf{C}^{\mathrm{iv}} = \begin{bmatrix} 494 & 158 & 114 & -23 & 0 & 0 \\ 158 & 494 & 114 & 23 & 0 & 0 \\ 114 & 114 & 496 & 0 & 0 & 0 \\ -23 & 23 & 0 & 145 & 0 & 0 \\ 0 & 0 & 0 & 0 & 145 & -23 \\ 0 & 0 & 0 & 0 & -23 & 168 \end{bmatrix} \mathrm{GPa}. \tag{20}$$

Boundary element solutions were obtained on three different meshes (see Figs. 4-6) with the 216 elements (mesh A), 384 elements (mesh B) and 600 elements (mesh C).

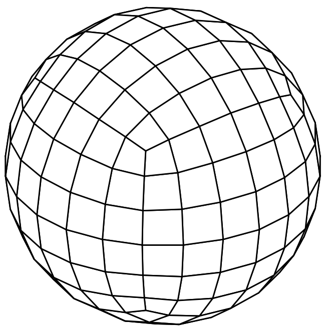


Fig. 4. Boundary element mesh A

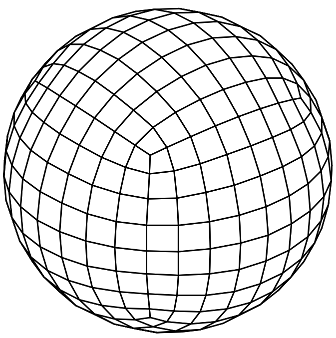


Fig. 5. Boundary element mesh B

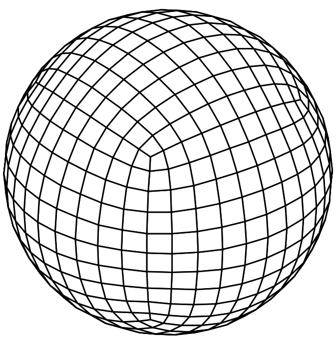


Fig. 6. Boundary element mesh C

The radial displacements $u_r = \sqrt{u_1^2 + u_2^2 + u_3^2}$ were studied. With the angles $-\pi \le \alpha < \pi$ and $-\pi \le \beta < \pi$ (see Fig. 1) defining the nodes respectively located along the circumference at the cavity surface on the plane $(x_1, x_2, 0)$ and on the plane $(0, x_2, x_3)$, the comparison of the obtained results with the corresponding results provided in [10] are given in Figs. 7-13.

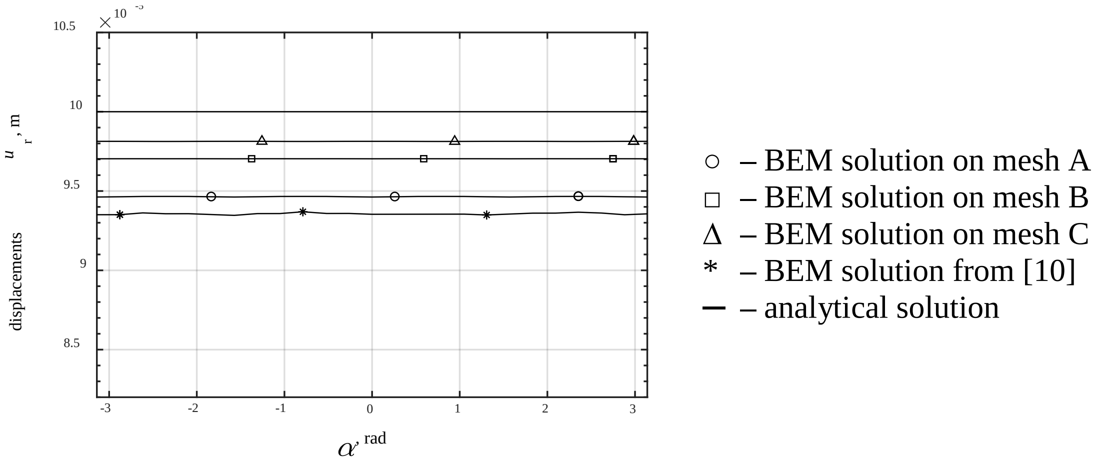


Fig. 7. Displacements u_r at the nodes on the plane x_1x_2 for aluminum polycrystal

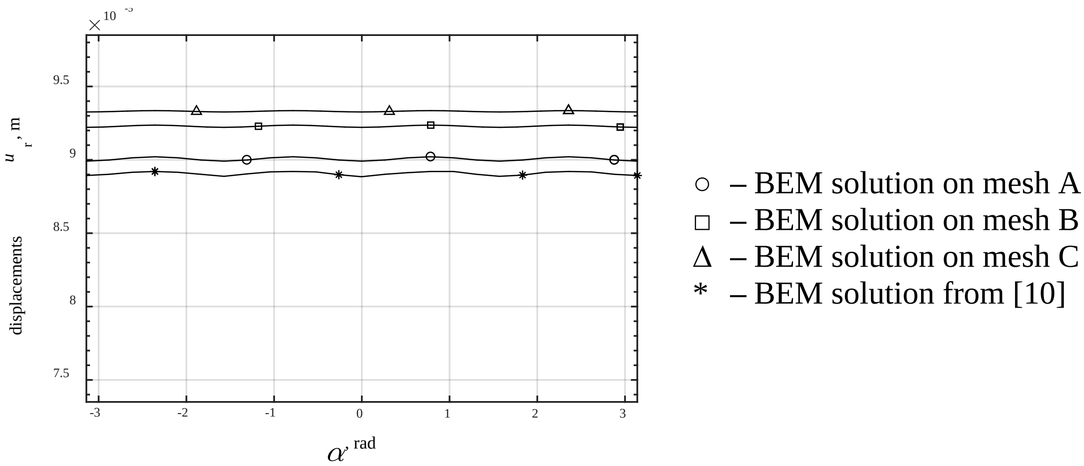


Fig. 8. Displacements u_r at the nodes on the plane x_1x_2 for aluminum crystal

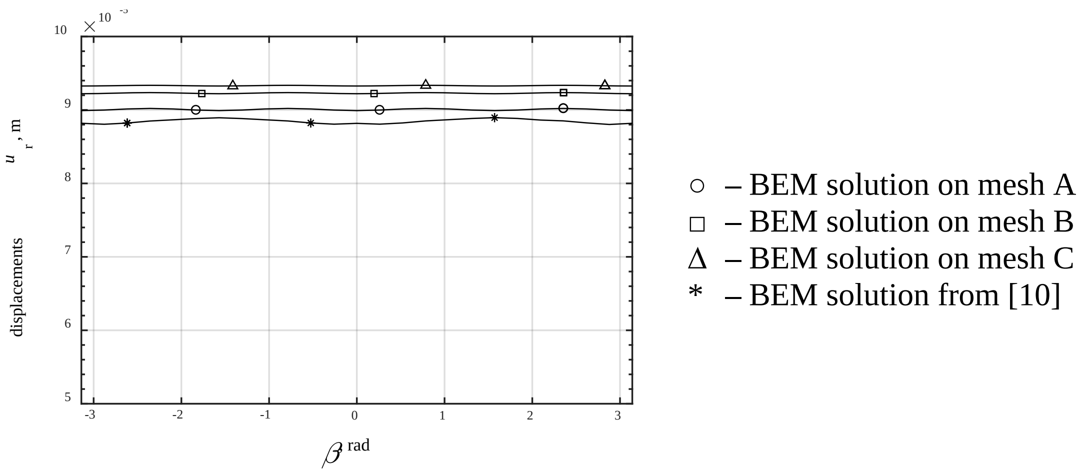


Fig. 9. Displacements u_r at the nodes on the plane x_2x_3 for aluminum crystal

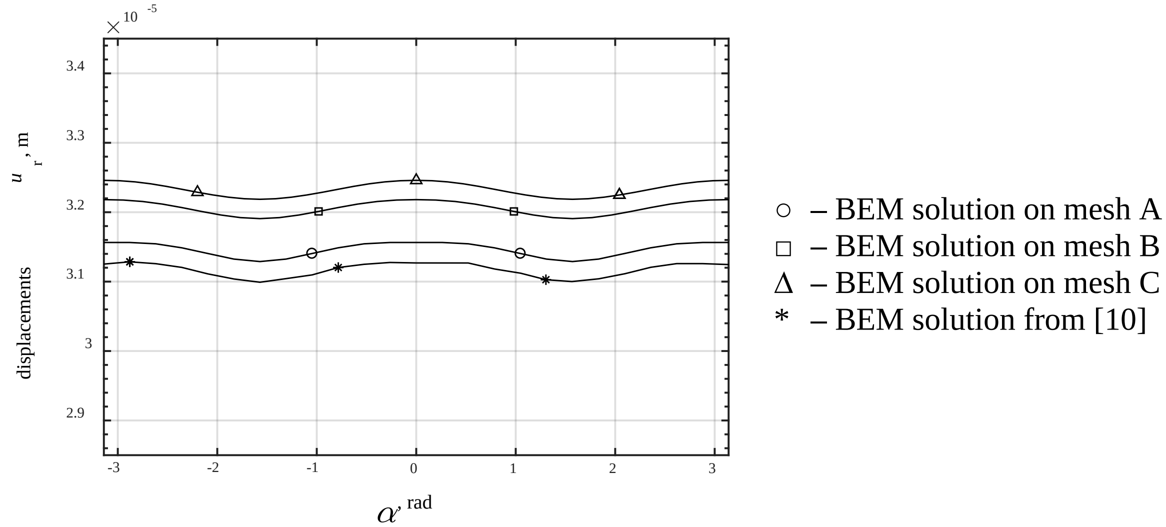


Fig. 10. Displacements u_r at the nodes on the plane x_1x_2 for barium sodium niobate

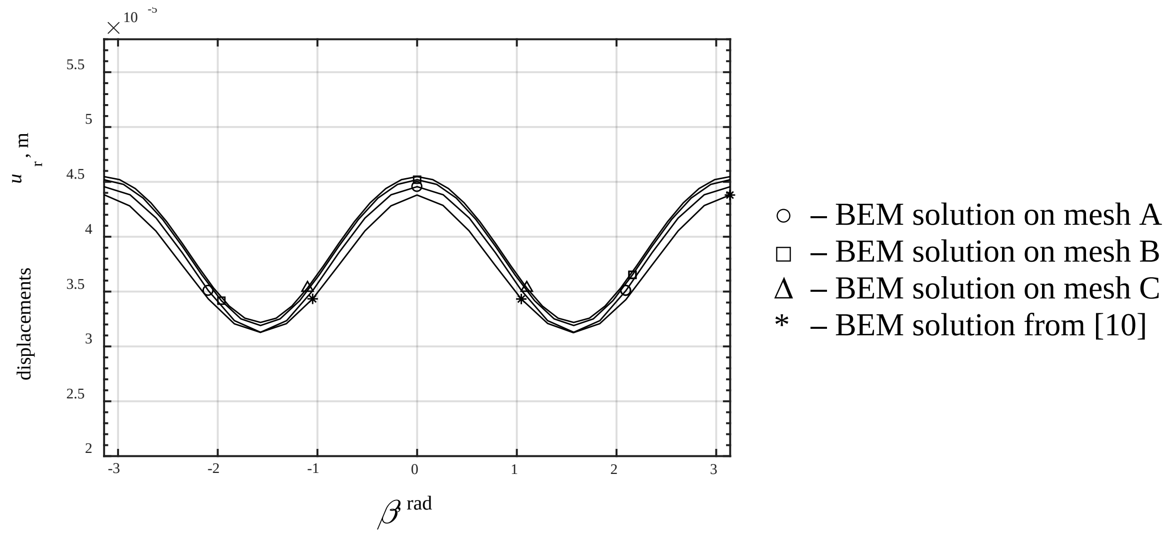


Fig. 11. Displacements u_r at the nodes on the plane x_2x_3 for barium sodium niobate

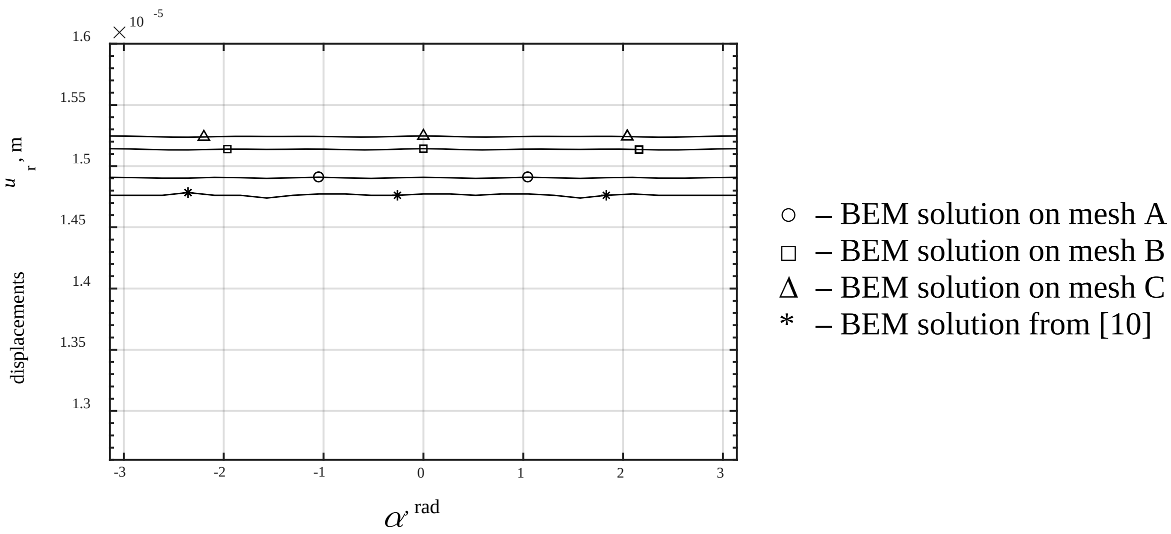


Fig. 12. Displacements u_r at the nodes on the plane x_1x_2 for sapphire

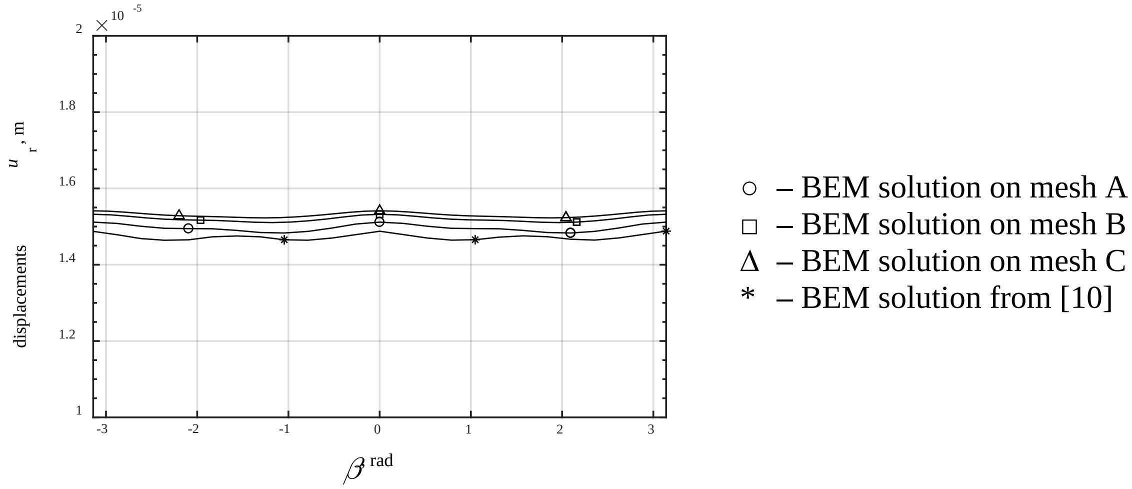


Fig. 13. Displacements u_r at the nodes on the plane x_2x_3 for sapphire

4. Conclusions

The boundary element formulation for static homogeneous anisotropic linearly elastic problems was presented in this work. The formulation is based on the weakly singular boundary integral equation and mixed representation of the geometry of the boundary elements and of boundary fields. For the efficient computation of the static anisotropic fundamental solutions an interpolation scheme was employed. Accuracy and reliability of the presented boundary element approach is demonstrated with the numerical example of an infinite elastic medium containing a spherical cavity under internal pressure. For four materials with different degrees of anisotropy the responses of radial displacements at the nodes located on the two coordinate planes were obtained. Convergence with the mesh refinement was established for all considered materials. Obtained solutions were found to be in a good agreement with analytical solutions and boundary element results provided in the literature.

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References

[1] Wang CY, Achenbach JD. Elastodynamic fundamental solutions for anisotropic solids. *Geophys. J. Int.* 1994;118(2): 384-392.
[2] Wang CY, Achenbach JD. Three-dimensional time-harmonic elastodynamic Green's functions for anisotropic solids. *Proc. R. Soc. Lond. A.* 1995;449(1937): 441-458.
[3] Wilson RD, Cruse TA. Efficient implementation of anisotropic three-dimensional boundary-integral equation stress analysis. *Int. J. Numer. Methods Eng.* 1978;12(9): 1383-1397.
[4] Vogel SM, Rizzo FJ. An integral equation formulation of threedimensional anisotropic elastostatic boundary value problems. *J. Elast.* 1973; 3(3): 203-216.
[5] Chen T, Lin FZ. Boundary integral formulations for three-dimensional anisotropic piezoelectric solids. *Computational Mechanics.* 1995;15(6): 485-496.
[6] Ting TCT, Lee VG. The three-dimensional elastostic Green's function for general anisotropic linear elastic solid. *Q. J. Mech. Appl. Math.* 1997;50(3): 407-426.
[7] Nakamura G, Tanuma K. A formula for the fundamental solution of anisotropic elasticity. *Q. J. Mech. Appl. Math.* 1997; 50(2): 179-194.
[8] Sales MA, Gray LJ. Evaluation of the anisotropic Green's function and its derivatives *Comput. Struct.* 1998;69(2): 247-254.

[9] Tonon F, Pan E, Amadei B. Green's functions and boundary element method formulation for 3D anisotropic media. *Comput. Struct.* 2001;79(5): 469-482.
[10] Wang CY, Denda M. 3D BEM for general anisotropic elasticity. *Int. J. Solids Struct.* 2007;44(22-23): 7073-7091.
[11] Shiah YC, Tan CL, Wang CY. Efficient computation of the Green's function and its derivatives for three-dimensional anisotropic elasticity in BEM analysis. *Eng. Anal. Boundary Elem.* 2012;36(12): 1746-1755.
[12] Tan CL, Shiah YC, Wang CY. Boundary element elastic stress analysis of 3D generally anisotropic solids using fundamental solutions based on Fourier series. *Int. J. Solids Struct.* 2013;50(16-17): 2701-2711.
[13] Bazhenov VG, Igumnov LA. *Boundary Integral Equations and Boundary Element Methods in Treating the Problems of 3D Elastodynamics with Coupled Fields.* Moscow: PhysMathLit; 2008. (In Russian)
[14] Ugodchikov AG, Hutoryanskii NM. *Boundary element method in deformable solid mechanics.* Kazan: Kazan State University Press; 1986. (In Russian)