

## ON A CONFIGURATIONAL FORCE DRIVING SURFACE GROWTH OF SOLIDS

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**Abstract.** Surface growth of solids includes a wide variety of processes starting from additive manufacturing technologies and ending with plenty of biological problems. Model of surface growth based on the expression of the configurational force, derived from fundamental laws and entropy inequality, is offered in the present paper. A new expression for the configurational force called *surface growth force*, which controls growth and resorption, is used. Analysis of the applicability of the developed model was carried out for problems of surface growth of compressed cylinder with prestrained growth layer and growth layer without prestrain. The influence of matter supply function and mechanical stresses as one of the main process controlling factors on the surface growth was also investigated.

**Keywords:** configurational force, growth, surface growth tensor, biomechanics

### 1. Introduction

Surface growth of solids is a process that involves many different phenomena. A wide variety of additive manufacturing technologies, geological growth such as rock formation or ice cover growth, gravitational accretion, planet growth, biological growth such as bone growth and remodeling, tumor growth, brain growth etc. are among them. During the surface growth process mass, volume or geometry of the growing body changes due to the addition or apposition of mass on its surface. Nowadays, numerous experiments show that the growth process can be controlled by mechanical stimuli. And not only growth of non-living matter could be controlled but also growth of living tissue, e.g. bone remodeling process, being adaptive to the environment process, could be controlled by mechanical and, moreover, electrical stimuli. In addition, from mechanical point of view, residual stresses or prestrain/prestress in the growth layer can occur during the growth process. All the aforesaid supports the fact that the growth problem has been one of the major challenge of mechanics during last several decades.

Some problems of growing solids based on the differential-geometrical methods were considered in various papers by Lychev and Manzhurov [1,2]. Some coverage of geometrical methods adopted for the mechanics of incompatible strains which arise due to the growing process can be found in works of Yavari [3]. Also mechanics of growing solids was extensively covered by Arutyunyan et al. [4]. A theory of material growth based on the irreversible thermodynamics, where the Eshelby stress is represented as the driving force of irreversible growth, was presented in the works of Epstein and Maugin, see [5,6] and also reference therein. Mathematical framework that describes kinematics of surface growth was extensively developed by Skalac et al. [7].

A large number of works is dedicated to biological growth. As was already said, a lot of biological tissues undergo surface growth. Mostly it takes part in hard tissues during bones, nails, horns and etc. formation and adaptation. Goriely [8] provides a general mathematical theory for biological growth and lots of examples and problems including outstanding ones. Approaches to growth driving forces, similar to the one used in the present paper, were developed by Ganghoffer et al. in [9,10]. In these works, a model in the framework of thermodynamics of irreversible phenomena was developed. The driving force for process of bone remodeling is identified as a surface divergence of Eshelby stress. Moreover, numerical simulation was presented in these works. In addition, surface growth plays an important role for soft tissues, for example the brain. A computational model predicting realistic surface morphologies of brain was developed by Kuhl et al. [11] using the continuum theory of finite growth. In these work theoretical, numerical, and experimental data are presented.

Surface growth is similar to the chemical reaction front propagation. Approach based on chemical affinity tensor which is an analogue to surface growth tensor presented in this paper was developed in the various papers of Freidin et al. [12,13]. From the mechanical point of view, in contrast to chemical reactions considered in the mentioned papers where the reaction front divides solid constituents, surface growth process is accompanied by the appearance of new material point on the surface of growing body.

However, despite the fact that the surface growth problem is one of the most important areas of mechanical research, there is still no commonly accepted model of the growth process. In present research surface growth tensor based on the mechanics of configurational forces, fundamental mechanical laws, and driving surface growth process is presented.

## 2. Configurations and kinematics

In the present paper surface growth is considered as the reaction on the surface of solid body  $B_* \rightarrow B_+$ ,

where solid part is indicated by the symbol (+), the diffusive one is indicated by the symbol (\*). New solid layers appear due to the transformation of diffusive matter into the solid on the surface. For example, in the case of biological growth there are generating cells on the surface, which are producing new material from the nutrients. Initial body consists of both solid and diffusive constituents. Diffusive matter can appear in the body due to various reasons depending on the process: income into the volume of body (volumetric supply) or diffusion through the boundary into the volume. Diffusive constituent (e.g. nutrients, in the case of biological growth or resorption) diffuses to the surface. Thus, there is a sink of (\*) constituent and source of (+) on the surface.

**Configurations.** Let  $v^t$  be the current configuration of the growing body at time  $t$  and  $V^0$  – the reference configuration, which depends on the history of previous growth. The outer surface of the growing body is growing surface  $\Gamma$ . In the present paper we use an assumption of the absence of additional deformations due to the diffusive constituent. Linear segments  $d\mathbf{x}_+$  of the current configuration are related to the corresponding ones in the reference configuration by the deformation gradient as

$$d\mathbf{x}_+ = \mathbf{F} \cdot d\mathbf{X}_+; \det \mathbf{F} = \frac{dv_+^t}{dV_+^0} = \frac{\rho_+^0}{\rho_+^t}, \quad (2)$$

where  $\rho_+^t$  is the partial density of solid constituent in the current configuration,  $\rho_+^0$  is the density of solid constituent per unit volume of the reference configuration. It follows that masses in the reference and current configurations are related by

$$\rho_+^t dv_+^t = \rho_+^0 dV_+^0. \quad (3)$$

For diffusive constituent there is the following relation

$$\rho_*^t dv_*^t = \rho_*^0 dV_*^0, \quad (4)$$

where  $\rho_*^t, \rho_*^0$  are the densities of diffusive component per unit volume of the current and corresponding unit volume of the reference configurations respectively.

Area elements  $d\omega$ ,  $d\Omega$  and unit normals  $\mathbf{n}$ ,  $\mathbf{N}$  in the current and reference configurations respectively are related by

$$\mathbf{n}d\omega = (\det \mathbf{F})\mathbf{F}^{-T} \cdot \mathbf{N}d\Omega. \quad (5)$$

**Kinematics.** Growth velocity  $\mathbf{v}_\Gamma^t$  in the current configuration is related to the growth velocity  $\mathbf{V}_\Gamma^0$  with respect to the reference configuration and velocity  $\mathbf{v}_+$  of material point  $B_+$  as

$$\mathbf{v}_\Gamma^t = \frac{d\mathbf{x}_+(\mathbf{X}_\Gamma^0(t), t)}{dt} = \mathbf{F} \cdot \mathbf{V}_\Gamma^0 + \mathbf{v}_+. \quad (6)$$

Moreover, one can assume that

$$\mathbf{V}_\Gamma^0 = W\mathbf{N}, \quad (7)$$

where  $W$  is the normal component of growth velocity.

For diffusive particles the velocity  $\mathbf{v}_*$  is introduced as an effective velocity of the particles smeared over the volume of the solid constituent. Then the Lagrangian diffusion velocity  $\mathbf{V}_*^0$  with respect to the reference configuration can be introduced as

$$\mathbf{V}_*^0 = \mathbf{F}^{-1} \cdot (\mathbf{v}_* - \mathbf{v}_+), \quad (8)$$

where  $(\mathbf{v}_* - \mathbf{v}_+)$  is the relative velocity that characterizes the diffusion in the current configuration, and

$$\mathbf{v}_* = \mathbf{F} \cdot \mathbf{V}_*^0 + \mathbf{v}_+. \quad (9)$$

More detailed description of configurations and kinematics can be found in the [14].

### 3. Balance equations

Balance equations and dissipation inequality for considering growing body were derived (see Appendix). Main equations are presented below.

General mass balance in the reference configuration is

$$\rho_+^0 W + \rho_*^0 (\mathbf{V}_*^{ex} \cdot \mathbf{N} + W - \mathbf{V}_*^{in} \cdot \mathbf{N}) = 0, \quad (10)$$

where  $\mathbf{V}_*^{in}$  is the diffusion velocity of particles which take part in the formation of growth layer,  $\mathbf{V}_*^{ex}$  is the velocity of particles leaving the body through the surface. If the whole diffusive constituent transforms into the solid, i.e. there is no particles leaving the body and  $\mathbf{V}_*^{ex} = 0$ , then

$$\rho_+^0 W = \rho_*^0 (\mathbf{V}_*^{in} \cdot \mathbf{N} - W). \quad (11)$$

General momentum balance in the reference configuration is

$$\hat{\rho}_+^0 \mathbf{v}_+ + \hat{\rho}_*^0 \mathbf{v}_*^{in} = -\llbracket \mathbf{S}_+ \rrbracket \cdot \mathbf{N} - \llbracket \mathbf{S}_* \rrbracket \cdot \mathbf{N} + \rho_*^0 (\mathbf{v}_*^{in} - \mathbf{v}_*^{ex}) \mathbf{V}_*^{ex} \cdot \mathbf{N}, \quad (12)$$

where  $\hat{\rho}_*^t$  is the surface sink of diffusive constituent,  $\hat{\rho}_+^t$  is the surface source of solid constituent,  $\mathbf{S}_{+,*}^{in/ex} = (\det \mathbf{F}) \boldsymbol{\sigma}_{+,*}^{in/ex} \cdot \mathbf{F}^{-T}$  are the Piola-Kirchhoff stresses determined with respect to the reference configuration and  $\llbracket \mathbf{S}_{+,*} \rrbracket = \mathbf{S}_{+,*}^{in} - \mathbf{S}_{+,*}^{ex}$ , and the last term refers to the diffusion process.

Energy balance takes the following form

$$\begin{aligned} & \hat{\rho}_+^0 \left( u_+ + \frac{1}{2} \mathbf{v}_+^2 \right) + \hat{\rho}_*^0 \left( u_* + \frac{1}{2} \mathbf{v}_*^{in2} \right) = \\ & = \rho_*^0 \llbracket \mathbf{v}_* \rrbracket \langle \mathbf{v}_* \rangle \mathbf{V}_*^{ex} \cdot \mathbf{N} + \mathbf{h}_0 \cdot \mathbf{N} + q_\Gamma - (\mathbf{v}_+ \cdot \llbracket \mathbf{S}_+ \rrbracket + \mathbf{v}_*^{in} \cdot \llbracket \mathbf{S}_* \rrbracket) \cdot \mathbf{N}, \end{aligned} \quad (13)$$

where  $u_{+,*}$  are the specific internal energies of constituents,  $\mathbf{h}_0$  is the Piola-Kirchhoff heat flux,  $q_\Gamma$  is the rate of surface heat supply.

Consequently, the surface density of the dissipation on the growing surface is

$$\begin{aligned} D = & - \left[ \hat{\rho}_+^0 \left( f_+ + \frac{1}{2} \mathbf{v}_+^2 \right) + \hat{\rho}_*^0 \left( f_* + \frac{1}{2} \mathbf{v}_*^{in2} \right) + (\mathbf{v}_+ \cdot \llbracket \mathbf{S}_+ \rrbracket + \mathbf{v}_*^{in} \cdot \llbracket \mathbf{S}_* \rrbracket) \cdot \mathbf{N} - \right. \\ & \left. - \rho_*^0 \llbracket \mathbf{v}_* \rrbracket \langle \mathbf{v}_* \rangle \mathbf{V}_*^{ex} \cdot \mathbf{N} \right] + (1 - \alpha) q_\Gamma \geq 0, \end{aligned} \quad (14)$$

where  $f_{+,*} = -T s_{+,*} + u_{+,*}$  are the densities of Helmholtz free energy.

#### 4. Surface growth tensor

From chemistry it is known that entropy production can be represented as

$$P[S] = A\omega, \quad (15)$$

where  $\omega$  is the reaction rate,  $A$  is the chemical affinity. The goal is to present Eq. (14) in the similar way.

Basing on the balance equations and the second law of thermodynamic, denoting Eshelby stress tensor as  $\mathbf{b}_+ = \rho_+ f_+ \mathbf{I} - \mathbf{F}_+^T \cdot \llbracket \mathbf{S}_+ \rrbracket = \rho_+ \mathbf{M}_+$  and chemical potential of diffusive constituent as  $\mu_* = f_* - \mathbf{N} \cdot \mathbf{F}_+^T \cdot \mathbf{S}_* \cdot \mathbf{N} / \rho_*$ , and assuming that the whole diffusive constituent transforms into the solid, the following equation for dissipation can be derived (see Appendix):

$$D = \frac{\rho_+}{M_+} W A_{NN} + (1 - \alpha) q_\Gamma = \omega (A_{NN} + \hat{q}) \geq 0, \quad (16)$$

where  $\hat{q}$  can be considered as the parameter of the reaction,  $A_{NN} = \mathbf{N} \cdot \mathbf{A} \cdot \mathbf{N}$  is the normal component of the surface growth tensor  $\mathbf{A}$ . The growth tensor takes the form similar to chemical affinity tensor in [12,13] and is equal to

$$\mathbf{A} = M_* \mu_* \mathbf{I} - M_+ \mathbf{M}_+ - \left( \frac{1}{2} (\mathbf{v}_+^2 - \mathbf{v}_*^{in2}) + \mathbf{v}_\Gamma^t (\mathbf{v}_*^{in} - \mathbf{v}_+) \right) \mathbf{I}. \quad (17)$$

In quasi-static case the surface growth tensor is

$$\mathbf{A} = M_* \mu_* \mathbf{I} - M_+ \mathbf{M}_+. \quad (18)$$

**Kinetics.** It can be shown (see [12]) that the reaction rate at the surface element is

$$\omega_N = \bar{\omega} \left( 1 - \exp \left( - \frac{A_{NN}}{RT} \right) \right), \quad (19)$$

and the growth velocity in the reference configuraton due to the mass balance is

$$W = \frac{M_+}{\rho_+} k_*^0 c^0 \left( 1 - \exp \left( - \frac{A_{NN}}{RT} \right) \right), \quad (20)$$

where  $k_*^0$  is the kinetic constant,  $c^0$  is the molar concentration of diffusive particles per unit volume.

In the case of equilibrium concentration  $c_{eq}$  there is no growth or resorption, so there is no reaction on the surface, and

$$A_{NN}(c = c_{eq}) = M_* \mu_*(c_{eq}, T) - M_+ \mu_+ = 0. \quad (21)$$

Finally, omitting some calculations which can be found in [12], growth velocity takes the following form:

$$W = \frac{M_+}{\rho_+} k_* (c(\Gamma) - c_{eq}), \quad (22)$$

where  $c(\Gamma)$  is the concentration on the growing surface. If  $c(\Gamma) > c_{eq}$  then  $W > 0$  and it corresponds to the surface growth, otherwise  $W < 0$  and it corresponds to the surface resorption.

**Diffusion.** Concentration  $c(\Gamma)$  on the growing surface can be found from the diffusion problem. Let us assume that diffusion flux is given by the Fick's law

$$\mathbf{j} = -D \nabla c, \quad (23)$$

where  $D$  is the diffusion coefficient. Then in steady state process and if  $D$  is constant the diffusion equation with source is

$$\Delta c + s = 0, \quad (24)$$

where  $s$  is a source term depending on stresses and body configuration.

Boundary condition on the growing surface comes from the mass balance so that all diffused to the surface matter transforms into the new one,

$$D \frac{\partial c}{\partial N} \Big|_\Gamma + \frac{\rho}{M} W = 0. \quad (25)$$

Thus, using Eq. (22), growth velocity can be found by finding  $c_{eq}$  from the Eq. (21) for normal component of the surface growth tensor and finding  $c(\Gamma)$  from the diffusion problem.

**Small strain approach.** Denoting Helmholtz free energy as  $w_+ = \rho_+^0 f_+$ , normal component of surface growth tensor in quasi-static is

$$A_{NN} = \frac{M_+}{\rho_+} (-w_+ + \mathbf{F}_+^T : \llbracket \mathbf{S}_+ \rrbracket) + M_* \mu_*. \quad (26)$$

In the case of small strains and linear-elastic constituent:

$$w_+ = \eta_+(T) + \frac{1}{2} (\boldsymbol{\varepsilon}_+ - \boldsymbol{\varepsilon}^{gr}) : \mathbf{C} : (\boldsymbol{\varepsilon}_+ - \boldsymbol{\varepsilon}^{gr}), \quad (27)$$

where  $\eta_+(T)$  is the free energy volume density of the stress-free solid constituent,  $\mathbf{C}$  is the elasticity tensor,  $\boldsymbol{\varepsilon}^{gr}$  is the growth strain (prestrain in the growth layer). Depending on the phenomenon, growth layer could be characterized by: zero stresses; stresses equal to those in the initial body; stresses with some value of the prestress. It is assumed that in the initial part of growing body prestrain is equal to zero. Prestrain and prestress could have a significant impact on the growth process speeding it up or slowing it down.

The Cauchy stress is related to strains as

$$\boldsymbol{\sigma}_+ = \mathbf{C} : (\boldsymbol{\varepsilon}_+ - \boldsymbol{\varepsilon}^{gr}). \quad (28)$$

Let us assume that chemical potential of diffusive constituent is equal to

$$M_* \mu_* = \eta_*(T) + RT \ln \left( \frac{c}{c_*^0} \right). \quad (29)$$

where  $c_*^0$  is the initial concentration.

Consequently, normal component of surface growth tensor within small-strain approach is

$$A_{NN} = \frac{M_+}{\rho_+} (\gamma(T) - 1/2 (\boldsymbol{\varepsilon}_+ - \boldsymbol{\varepsilon}^{gr}) : \mathbf{C}_+ : (\boldsymbol{\varepsilon}_+ - \boldsymbol{\varepsilon}^{gr}) + \boldsymbol{\varepsilon}_+ : \llbracket \boldsymbol{\sigma}_+ \rrbracket) + RT \ln \frac{c}{c_*^0}, \quad (30)$$

where  $\gamma(T) = -\eta_+(T) + \eta_*(T)\rho_+/M_+$  is the input of reaction energies (a parameter of the model).

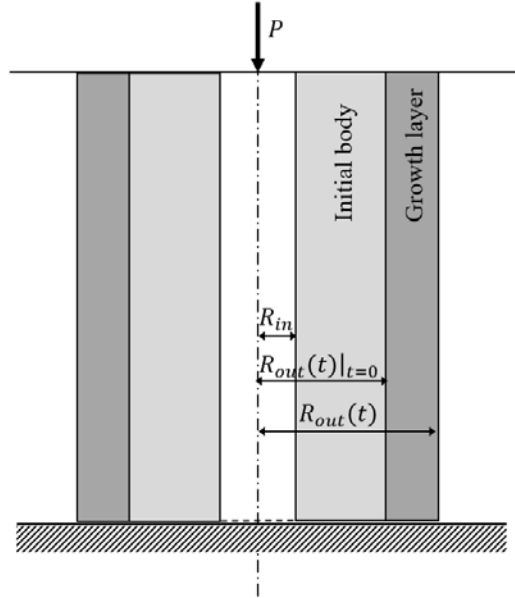
## 5. Surface growth of the compressed cylinder

The developed model was applied to the problem of surface growth of a cylinder under compression. Growth of trees or remodeling of bone are the examples of such problem. In the case of bone this process mainly takes place in the trabecular part, and under the applied stress surface growth or resorption occurs on the surfaces of trabeculae (small tissue element that has a form similar to cylinder).

In the following problems a cylinder under axial load is considered. Growth is controlled by the normal component of surface growth tensor that depends on stresses/strains and the concentration of the diffusive matter. Supplying function of diffusive matter (the volume source) is a function of stresses. Stresses in turn depend on the current body configuration. It is assumed that there are certain equilibrium stresses depending on the body configuration when no growth happens. It is supposed that supplying function is equal to zero in the equilibrium state which corresponds to equilibrium stresses, positive in the case of growth and negative in the case of resorption. Therefore, this provides a coupling between stresses, growth rate and the matter supply.

Based on the examples mentioned above, it is assumed that growth takes place only at the lateral surfaces of the cylinder. Two cases of surface growth were considered. The first one is without prestrain in the growth layer. The second one – presence of growth strain depending on radial coordinate and on current outer radius.

**Surface growth without prestrain.** Infinite cylinder with a hollow of radius  $R_{in}$  under axial load  $P$  is considered (see Fig. 1). Current outer radius  $R_{out}$  changes over time, increases in the case of growth and decreases in the case of resorption.



**Fig. 1.** Growing cylinder under axial load  $P$

Axial stress is

$$\sigma_z = \frac{P}{\pi(R_{out}^2 - R_{in}^2)}. \quad (31)$$

To find the concentration at the growing surface  $c(R_{out})$  one have to solve the following diffusion problem

$$\frac{\partial^2 c(r)}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial c(r)}{\partial r} + \frac{\hat{s}(\sigma_z)}{D \cdot M} = 0, \quad (32)$$

where  $\hat{s}(\sigma_z)$  is supplying function depending on axial stress. Boundary conditions are

$$\begin{aligned} \left. \frac{\partial c(r)}{\partial r} \right|_{r=R_{in}} &= 0; \\ -D \left. \frac{\partial c(r)}{\partial r} \right|_{r=R_{out}} &= k_* (c(R_{out}) - c_{eq}). \end{aligned} \quad (33)$$

After substitution stresses into the Eq. (30) the equation for equilibrium concentration  $c_{eq}$  can be found

$$c_{eq} = c_*^0 \exp\left(-\frac{\gamma M}{RT\rho} + \frac{P^2 M}{4E\pi^2 R(R_{out}^2 - R_{in}^2)^2 T\rho}\right). \quad (34)$$

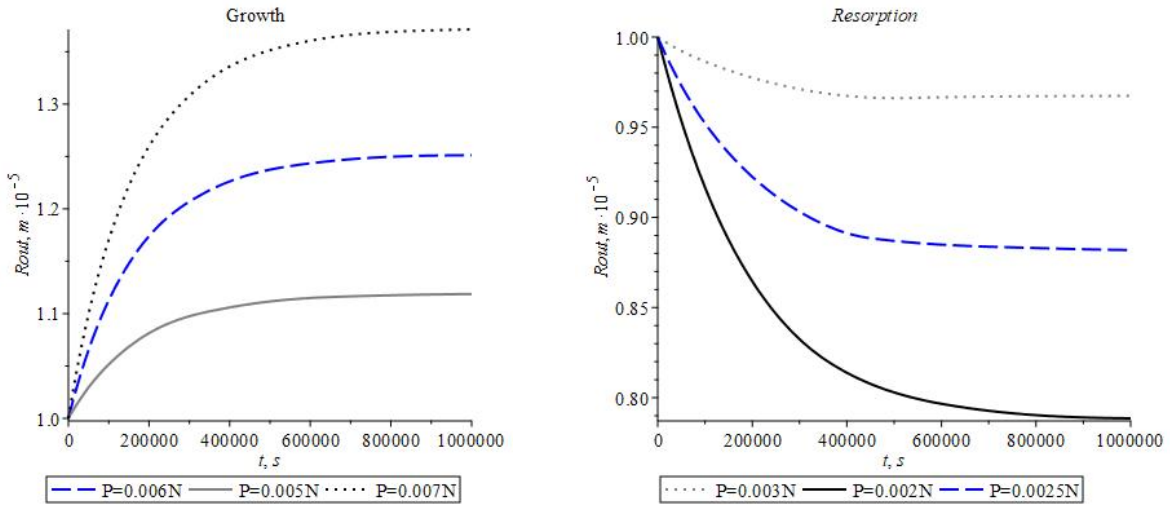
Then after finding  $c_{eq}$  and  $c(R_{out})$  the reaction front velocity can be found from the following equation

$$W = \frac{dR_{out}(t)}{dt} = \frac{M_+}{\rho_+} k_* (c(R_{out}) - c_{eq}). \quad (35)$$

Supplying function which controls the amount of the matter coming into the volume depends on stresses. Three different dependence on stresses were considered: linear, hyperbolic sinus and hyperbolic tangent dependence. The dependences were chosen in such way that supplying function is equal to zero at equilibrium stresses.

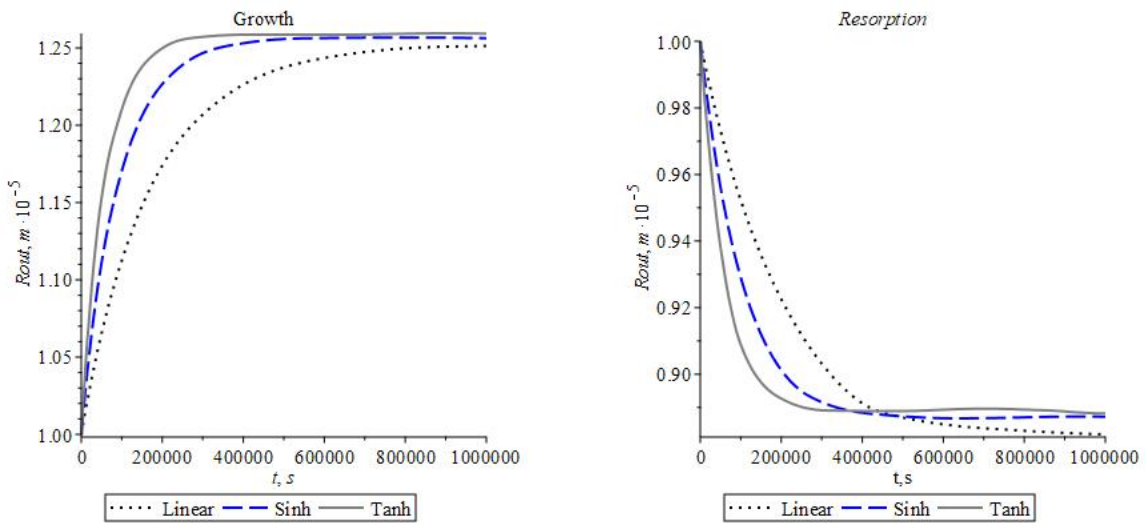
Bone tissue parameters were taken for further calculations [15]. And it was assumed that  $R_{in} \ll R_{out}$ . The following plots (see Fig. 2) were obtained for cylinder under compression by various forces. When the force is smaller than necessary for maintaining equilibrium state then resorption begins. Otherwise – growth. After a certain time (several days in the considered problem), a new equilibrium state for the applied force is achieved.

I.e. cylinder grows up to a radius that ensures that under applied force stresses become equal to those necessary for equilibrium.



**Fig. 2.** Dependence of the outer radius on time for various applied force: growth (left), resorption (right)

Growth and resorption for various supplying function under the same force are presented in the following plots (see Fig. 3). The magnitude of the outer radius reaches the same value for each supply but at different times.



**Fig. 3.** Dependence of the outer radius on time for various supply function: growth (left), resorption (right)

**Surface growth with prestrained growth layer.** In this case the same cylinder is considered. In this way applied stress is

$$\sigma_z = \frac{P}{\pi(R_{out}^2 - R_{in}^2)}. \tag{36}$$

The difference between the present and the previous cases is the presence of prestrain in the growth layer. Prestrain was chosen as follows

$$\varepsilon_{\varphi}^{gr} = -\varepsilon_r^{gr} = k^{gr} \cdot \ln \frac{r}{R_0}, \tag{37}$$

where  $R_0$  is the initial outer radius of the cylinder,  $k^{gr}$  is a prestrain coefficient.

To find strains and stresses in cylinder the following equilibrium equations have to be solved:

$$\nabla \cdot (C \cdot \varepsilon^e) = 0, r < R_0; \tag{38}$$

$$\nabla \cdot (C \cdot \varepsilon^e) = \nabla \cdot (C \cdot \varepsilon^{gr}), r \geq R_0, \tag{39}$$

with the following boundary conditions:

$$u_1(0) < \infty;$$

$$u_1(r = R_0) = u_2(r = R_0); \tag{40}$$

$$\sigma_{1r}(r = R_0) = \sigma_{2r}(r = R_0);$$

$$\sigma_r(r = R_{out}) = 0,$$

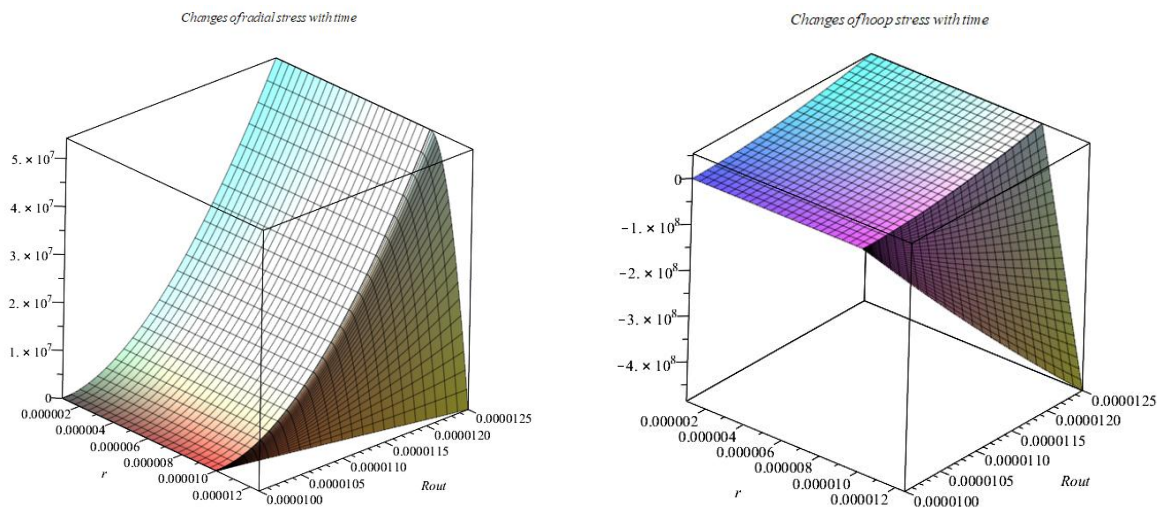
where index 1 indicates the initial part of the body and index 2 indicates the growing part.

For isotropic body and for  $R_{in} \ll R_{out}$  the following radial and hoop stresses were obtained.

$$\sigma_r = \begin{cases} \frac{k^{gr} \cdot \mu(\mu + \lambda)}{2\mu + \lambda} \ln^2 \frac{R_{out}}{R_0} & r < R_0, \\ \frac{k^{gr} \cdot \mu}{2\mu + \lambda} \ln^2 \frac{R_{out}}{R_0} - \frac{2k^{gr} \cdot \mu(\mu + \lambda)}{2\mu + \lambda} \ln^2 \frac{r}{R_0} & r \geq R_0; \end{cases} \tag{41}$$

$$\sigma_{\varphi} = \begin{cases} \frac{k^{gr} \cdot \mu(\mu + \lambda)}{2\mu + \lambda} \ln^2 \frac{R_{out}}{R_0} & r < R_0, \\ \frac{k^{gr} \cdot \mu}{2\mu + \lambda} \ln^2 \frac{R_{out}}{R_0} - \frac{2k^{gr} \cdot \mu(\mu + \lambda)}{2\mu + \lambda} \ln^2 \frac{r}{R_0} - \frac{4k^{gr} \cdot \mu(\mu + \lambda)}{2\mu + \lambda} \ln \frac{r}{R_0} & r \geq R_0. \end{cases} \tag{42}$$

Changes of stress over the radial coordinate of cylinder with time (outer radius as an analogue) are shown in the following plots (see Fig. 4).



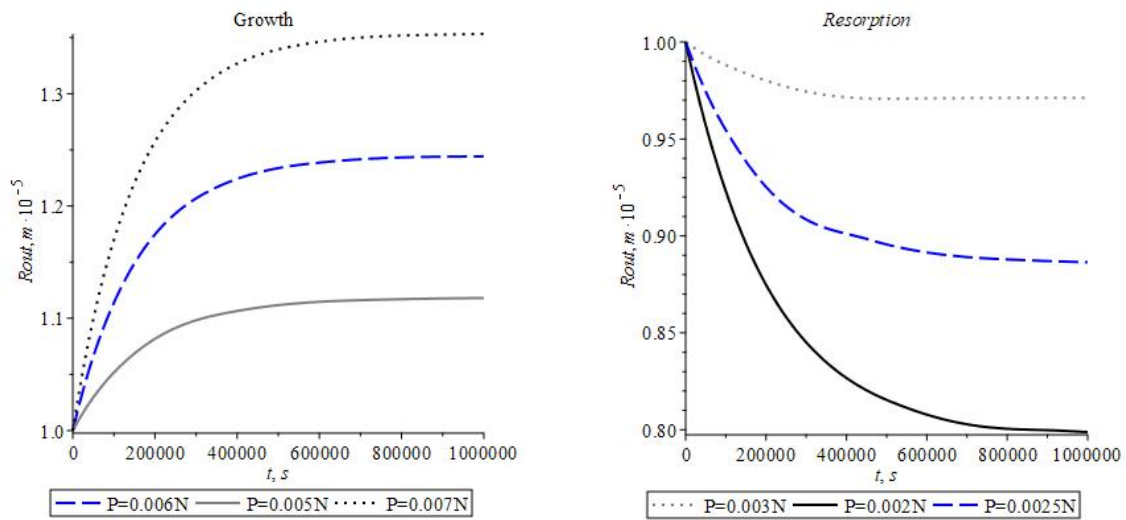
**Fig. 4.** Radial (left) and hoop (right) stresses distribution in cylinder versus time



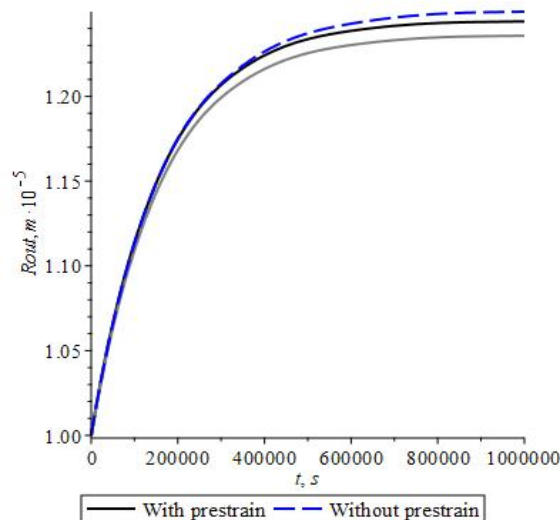
Diffusion problem for the considered case can be written as

$$\frac{\partial^2 c(r)}{\partial r^2} + \frac{1}{r} \frac{\partial c(r)}{\partial r} + \frac{\hat{s}(\sigma(r), r)}{D \cdot M} = 0. \tag{43}$$

Boundary conditions for diffusion problem are the same as in the previous case. However, since supply function in this case depends on the radial coordinate, the solution of the diffusion problem is more complex than that in previous case. In a way, similar to that used for the previous problem, equilibrium concentration and concentration on the boundary were obtained to get reaction front velocity. The following plots (see Fig. 5) represent dependence of the outer radius on time for the same forces as for the previous problem. In Figure 6 different prestrain coefficients were considered, and a comparison with the state without prestrain is given. It can be seen from Fig. 6 that prestrain changes the value of radius when the growth slows down.



**Fig. 5.** Dependence of the outer radius on time for various applied force: growth (left), resorption (right)



**Fig. 6.** Dependence of the outer radius on time for prestrained growth layer and for growth layer without prestrain

### 6. Conclusions

An approach to modeling stress-controlled surface growth basing on the surface growth tensor was developed. The expression for surface growth tensor was obtained from the balance laws and the second law of thermodynamics. The developed approach was applied for the problem of surface growth of a cylinder. It should be noted that the proposed model allows for prestrain in the growth layer to be taken into account. Moreover, the influence of the supplying function on the growth process was also considered.

The developed approach may be a proper frame for description of various problems including biological surface growth and resorption. Further research is going to be dedicated to more detailed study of the nature of prestrain and rejection of solid skeleton approach.

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### Appendix

Derivation of balance equations, as well as surface growth tensor are presented below.

**Mass balance.** Transport formulas in the current configuration are

$$\frac{Dm_+^t}{Dt} = \int_{V^t} \left( \frac{\partial \rho_+^t}{\partial t} + \nabla \cdot (\rho_+^t \mathbf{v}_+) \right) dV + \int_{\Gamma^t} \rho_+^t (w - \mathbf{v}_+ \cdot \mathbf{n}) d\omega; \tag{44}$$

$$\frac{Dm_*^t}{Dt} = \int_{V^t} \left( \frac{\partial \rho_*^t}{\partial t} + \nabla \cdot (\rho_*^t \mathbf{v}_+) \right) dV + \int_{\Gamma^t} \rho_*^t (w - \mathbf{v}_+ \cdot \mathbf{n}) d\omega. \tag{45}$$

In the reference configuration due to equations 3, 4:

$$m_0 = m_+ + m_*; \rho_0^0 = \rho_+^0 + \rho_*^0, \tag{46}$$

$$m_+^0 = \int_V \rho_+^0 dV; m_*^0 = \int_V \rho_*^0 dV. \tag{47}$$

Transport formulas in the reference configuration take the following form

$$\frac{Dm_+}{Dt} = \int_V \left( \frac{\partial \rho_+^0}{\partial t} + \nabla \cdot (\rho_+^0 \mathbf{v}_+) \right) dV + \int_{\Gamma^0} \rho_+^0 W d\Gamma; \tag{48}$$

$$\frac{Dm_*}{Dt} = \int_V \left( \frac{\partial \rho_*^0}{\partial t} + \nabla \cdot (\rho_*^0 \mathbf{v}_+) \right) dV + \int_{\Gamma^0} \rho_*^0 W d\Gamma. \tag{49}$$

The causes of the mass changes are the volumetric supply of diffusive constituent  $\mu_*^t$ , diffusion of particles to the surface with velocity  $v_*^{in}$ , diffusion of particles leaving the body through the surface with velocity  $v_*^{ex}$ , reaction on the surface with surface sink  $\hat{\rho}_*^t$  and surface source  $\hat{\rho}_+^t$ . It should be noted that only diffusive particles with velocity  $v_*^{in}$  transforms into the solid constituent. Thus, the following expressions can be written

$$\frac{Dm_+}{Dt} = \int_{\Gamma^t} \hat{\rho}_+^t d\Gamma; \tag{50}$$

$$\frac{Dm_*}{Dt} = \int_{\Gamma^t} \hat{\rho}_*^t d\Gamma + \int_{V^t} \mu_*^t dV - \int_{\partial V^t \setminus \Gamma^t} \rho_*^t (\mathbf{v}_*^{in} - \mathbf{v}_+) \cdot \mathbf{n} d\omega - \int_{\Gamma^t} \rho_*^t (\mathbf{v}_*^{ex} - \mathbf{v}_\Gamma^t) \cdot \mathbf{n} d\Gamma. \tag{51}$$

Taking into account that surface growth is under consideration, balances on the surface is going to be presented below. Using transport formulas and Eq. (50) and Eq. (51) one can obtain

$$\rho_+^t(w - \mathbf{v}_+ \cdot \mathbf{n}) = \hat{\rho}_+^t; \quad (52)$$

$$\rho_*^t(\mathbf{v}_*^{ex} \cdot \mathbf{n} - \mathbf{v}_*^{in} \cdot \mathbf{n}) = \hat{\rho}_*^t. \quad (53)$$

Then in the reference configuration mass balances take the form

$$\rho_+^0 W = \hat{\rho}_+^0; \quad (54)$$

$$\rho_*^0(\mathbf{V}_*^{ex} \cdot \mathbf{N} + W - \mathbf{V}_*^{in} \cdot \mathbf{N}) = \hat{\rho}_*^0. \quad (55)$$

**Momentum balance.** For momentum balance transport formulas in the current configuration are

$$\frac{D}{Dt} \int_{V^t} \rho_+^t \mathbf{v}_+ dV = \int_{V^t} \left( \frac{\partial \rho_+^t \mathbf{v}_+}{\partial t} + \nabla \cdot (\mathbf{v}_+ (\rho_+^t \mathbf{v}_+)) \right) dV + \int_{\Gamma^t} \rho_+^t \mathbf{v}_+ (w - \mathbf{v}_+ \cdot \mathbf{n}) d\Gamma; \quad (56)$$

$$\frac{D}{Dt} \int_{V^t} \rho_*^t \mathbf{v}_* dV = \int_{V^t} \left( \frac{\partial \rho_*^t \mathbf{v}_*^{in}}{\partial t} + \nabla \cdot (\mathbf{v}_+ (\rho_*^t \mathbf{v}_*^{in})) \right) dV + \int_{\Gamma^t} \rho_*^t \mathbf{v}_*^{in} (w - \mathbf{v}_+ \cdot \mathbf{n}) d\Gamma. \quad (57)$$

The causes of the momentum changes for solid and diffusive constituents are

$$\begin{aligned} \frac{D}{Dt} \int_{V^t} \rho_+^t \mathbf{v}_+ dV &= \underbrace{\int_{V^t} \rho_+^t \mathbf{b}_+ dV}_{\text{Body forces}} + \underbrace{\int_{\partial V^t \setminus \Gamma^t} \boldsymbol{\sigma}_+^{in} \cdot \mathbf{n} d\omega + \int_{\Gamma^t} \boldsymbol{\sigma}_+^{ex} \cdot \mathbf{n} d\Gamma}_{\text{Surface forces}} + \underbrace{\int_{\Gamma^t} \hat{\rho}_+^t d\Gamma + \int_{V^t} \hat{\pi}_+^t dV}_{\text{Supply due to growth}}; \quad (58) \end{aligned}$$

$$\begin{aligned} \frac{D}{Dt} \int_{V^t} \rho_*^t \mathbf{v}_* dV &= \underbrace{\int_{V^t} \rho_*^t \mathbf{b}_* dV}_{\text{Body forces}} + \underbrace{\int_{\partial V^t \setminus \Gamma^t} \boldsymbol{\sigma}_*^{in} \cdot \mathbf{n} d\omega + \int_{\Gamma^t} \boldsymbol{\sigma}_*^{ex} \cdot \mathbf{n} d\Gamma}_{\text{Surface forces}} + \underbrace{\int_{\Gamma^t} \hat{\rho}_*^t d\Gamma + \int_{V^t} \hat{\pi}_*^t dV}_{\text{Supply due to growth}} \\ &- \underbrace{\int_{\partial V^t \setminus \Gamma^t} \rho_*^t \mathbf{v}_*^{in} \otimes (\mathbf{v}_*^{in} - \mathbf{v}_+) \cdot \mathbf{n} d\omega - \int_{\Gamma^t} \rho_*^t \mathbf{v}_*^{ex} \otimes \rho_*^t (\mathbf{v}_*^{ex} - \mathbf{v}_\Gamma^t) \cdot \mathbf{n} d\Gamma}_{\text{Diffusion terms}} \end{aligned} \quad (59)$$

where  $\boldsymbol{\sigma}_+^{in}$  is the Cauchy stress tensor on inner side of the growing surface,  $\boldsymbol{\sigma}_+^{ex}$  is the Cauchy stress tensor on outer side of the growing surface including prestresses in the growth layer.

Then, taking into account mass balances, one can obtain momentum balances on the surface for each constituent

$$\begin{aligned} \hat{\rho}_+^t \mathbf{v}_+ &= -\boldsymbol{\sigma}_+^{in} \cdot \mathbf{n} + \boldsymbol{\sigma}_+^{ex} \cdot \mathbf{n} + \hat{\rho}_+^t; \\ \hat{\rho}_*^t \mathbf{v}_*^{in} &= -\boldsymbol{\sigma}_*^{in} \cdot \mathbf{n} + \boldsymbol{\sigma}_*^{ex} \cdot \mathbf{n} + \hat{\rho}_*^t + \rho_*^t \mathbf{v}_*^{in} (\mathbf{v}_*^{in} - \mathbf{v}_+) - \rho_*^t \mathbf{v}_*^{ex} (\mathbf{v}_*^{ex} - \mathbf{v}_\Gamma^t) \cdot \mathbf{n} d\Gamma. \end{aligned} \quad (60)$$

In the reference configuration:

$$\rho_+^0 \mathbf{v}_+ W = -\mathbf{S}_+^{in} \cdot \mathbf{N} + \mathbf{S}_+^{ex} \cdot \mathbf{N} + \hat{\rho}_+^0; \quad (61)$$

$$\rho_*^0 \mathbf{v}_*^{in} W = -\mathbf{S}_*^{in} \cdot \mathbf{N} + \mathbf{S}_*^{ex} \cdot \mathbf{N} + \hat{\rho}_*^0 + \rho_*^0 \mathbf{v}_*^{in} \otimes \mathbf{V}_*^{in} \cdot \mathbf{N} - \rho_*^0 \mathbf{v}_*^{ex} \otimes \mathbf{V}_*^{ex} \cdot \mathbf{N},$$

where  $\mathbf{S}_{+,*}^{in/ex} = (\det \mathbf{F}) \boldsymbol{\sigma}_{+,*}^{in/ex} \cdot \mathbf{F}^{-T}$  are the Piola-Kirchhoff stresses determined with respect to the reference configuration. Then, taking into account the mass balances, momentum balances take the following form

$$\hat{\rho}_+^0 \mathbf{v}_+ = (\mathbf{S}_+^{ex} - \mathbf{S}_+^{in}) \cdot \mathbf{N} + \hat{\rho}_+^0; \quad (62)$$

$$\hat{\rho}_*^0 \mathbf{v}_*^{in} = (\mathbf{S}_*^{ex} - \mathbf{S}_*^{in}) \cdot \mathbf{N} + \rho_*^0 (\mathbf{v}_*^{in} - \mathbf{v}_*^{ex}) \mathbf{V}_*^{ex} \cdot \mathbf{N} + \hat{\rho}_*^0. \quad (63)$$

As total momentum production is equal to zero  $\hat{p}_+^0 + \hat{p}_*^0 = 0$ , general momentum balance in the reference configuration is

$$\hat{\rho}_+^0 \mathbf{v}_+ + \hat{\rho}_*^0 \mathbf{v}_*^{in} = -\llbracket \mathbf{S}_+ \rrbracket \cdot \mathbf{N} - \llbracket \mathbf{S}_* \rrbracket \cdot \mathbf{N} + \rho_*^0 (\mathbf{v}_*^{in} - \mathbf{v}_*^{ex}) \mathbf{V}_*^{ex} \cdot \mathbf{N}. \quad (64)$$

**Energy balance.** Equations for energy balances are derived in the same way as previous balances. The causes of the kinetic and internal energy changes in the reference configuration are

$$\begin{aligned} \frac{D(K_+ + K_* + U_+ + U_*)}{Dt} &= \int_V (\rho_+^0 r_+ + \rho_*^0 r_* + \rho_+^0 \mathbf{b}_+ \cdot \mathbf{v}_+ + \rho_*^0 \mathbf{b}_* \cdot \mathbf{v}_*^{in} + \hat{u}_+ + \hat{u}_*) dV + \\ &+ \int_{\partial V \setminus \Gamma^0} \left( (\mathbf{v}_+ \cdot \mathbf{S}_+^{in} + \mathbf{v}_*^{in} \cdot \mathbf{S}_*^{in}) \cdot \mathbf{N} - \rho_*^0 u_* \mathbf{V}_*^{in} \cdot \mathbf{N} - \frac{1}{2} \rho_*^0 \mathbf{v}_*^{in2} \mathbf{V}_*^{in} \cdot \mathbf{N} \right) d\Omega + \\ &+ \int_{\Gamma^0} \left[ (\mathbf{v}_+ \cdot \mathbf{S}_+^{ex} + \mathbf{v}_*^{in} \cdot \mathbf{S}_*^{ex} - \rho_*^0 u_* \mathbf{V}_*^{ex} - \frac{1}{2} \rho_*^0 \mathbf{v}_*^{ex2} \mathbf{V}_*^{ex} + \mathbf{h}_0) \cdot \mathbf{N} + \hat{E}_+ + \hat{E}_* + q_\Gamma \right] d\Gamma, \end{aligned} \quad (65)$$

where  $r_{+,*}$  are the volumetric energy supply,  $\hat{u}_{+,*}$  are the volumetric energy production due to the growth,  $u_{+,*}$  are the specific internal energies of constituents,  $\hat{E}_+ + \hat{E}_*$  is the surface energy production due to growth,  $\mathbf{h}_0$  is the Piola-Kirchhoff heat flux,  $q_\Gamma$  is the rate of surface heat supply.

Then energy balance on the surface is

$$\begin{aligned} \rho_+^0 u_+ W + \frac{1}{2} \rho_+^0 \mathbf{v}_+^2 W + \rho_*^0 u_* W + \frac{1}{2} \rho_*^0 \mathbf{v}_*^{in2} W &= \mathbf{v}_+ \cdot \mathbf{S}_+^{ex} \cdot \mathbf{N} + \mathbf{v}_*^{in} \cdot \mathbf{S}_*^{ex} \cdot \mathbf{N} - \\ - \mathbf{v}_+ \cdot \mathbf{S}_+^{in} \cdot \mathbf{N} - \mathbf{v}_*^{in} \cdot \mathbf{S}_*^{in} \cdot \mathbf{N} - \rho_*^0 u_* \mathbf{V}_*^{ex} \cdot \mathbf{N} + \rho_*^0 u_* \mathbf{V}_*^{in} \cdot \mathbf{N} - \frac{1}{2} \rho_*^0 \mathbf{v}_*^{ex2} \mathbf{V}_*^{ex} \cdot \mathbf{N} + \\ + \frac{1}{2} \rho_*^0 \mathbf{v}_*^{in2} \mathbf{V}_*^{in} \cdot \mathbf{N} + \hat{E}_+ + \hat{E}_* + \mathbf{h}_0 \cdot \mathbf{N} + q_\Gamma. \end{aligned} \quad (66)$$

Taking into account Eqs. (54, 55, 62, 63) and that sum of energy productions is equal to zero  $\hat{E}_+^0 + \hat{E}_*^0 = 0$ , then energy balance takes following form

$$\begin{aligned} \hat{\rho}_+^0 \left( u_+ + \frac{1}{2} \mathbf{v}_+^2 \right) + \hat{\rho}_*^0 \left( u_* + \frac{1}{2} \mathbf{v}_*^{in2} \right) &= \\ = \rho_*^0 \llbracket \mathbf{v}_* \rrbracket (\mathbf{v}_*) \mathbf{V}_*^{ex} \cdot \mathbf{N} + \mathbf{h}_0 \cdot \mathbf{N} + q_\Gamma - \mathbf{v}_+ \cdot \llbracket \mathbf{S}_+ \rrbracket \cdot \mathbf{N} - \mathbf{v}_*^{in} \cdot \llbracket \mathbf{S}_* \rrbracket \cdot \mathbf{N}. \end{aligned} \quad (67)$$

**Entropy inequality.** The expressions for entropy  $S$  and entropy flux  $\Psi[S]$  in the reference configuration are

$$S = \int_V \rho_k^0 s_k dV; \quad (68)$$

$$\Psi[S] = \int_{\Gamma^0} \left( -\frac{\mathbf{h}_0 \cdot \mathbf{N}}{T} + \alpha \frac{q_\Gamma}{T} \right) d\Gamma + \int_V \frac{\rho_+^0 r_+ + \rho_*^0 r_*}{T} dV - \int_{\partial V \setminus \Gamma^0} \rho_*^0 s_* \mathbf{V}_*^{in} \cdot \mathbf{N} d\Omega - \quad (69)$$

$$- \int_{\Gamma^0} \rho_*^0 s_* \mathbf{V}_*^{ex} \cdot \mathbf{N} d\Gamma,$$

where  $s_k$  are the specific entropies.

Entropy production  $P[S] = P_V + P_\Gamma$  due to the second law of thermodynamics is

$$P[S] = \frac{dS}{dt} - \Psi[S] \geq 0. \quad (70)$$

From energy balance it follows that

$$\mathbf{h}_0 \cdot \mathbf{N} = \hat{\rho}_+^0 \left( u_+ + \frac{1}{2} \mathbf{v}_+^2 \right) + \hat{\rho}_*^0 \left( u_* + \frac{1}{2} \mathbf{v}_*^{in2} \right) - q_\Gamma - \rho_*^0 \llbracket \mathbf{v}_* \rrbracket \langle \mathbf{v}_* \rangle \mathbf{V}_*^{ex} \cdot \mathbf{N} + \mathbf{v}_+ \cdot \llbracket \mathbf{S}_+ \rrbracket \cdot \mathbf{N} + \mathbf{v}_*^{in} \cdot \llbracket \mathbf{S}_* \rrbracket \cdot \mathbf{N}. \quad (71)$$

Excluding flux term  $\mathbf{h}_0$  we obtain Clausius-Duhem inequality in a form

$$TP_\Gamma = \int_{\Gamma^0} D\Gamma \geq 0, \quad (72)$$

where surface density of the dissipation on the growing surface is

$$D = T(\hat{\rho}_+^0 s_+ + \hat{\rho}_*^0 s_*) - \hat{\rho}_+^0 \left( u_+ + \frac{1}{2} \mathbf{v}_+^2 \right) - \hat{\rho}_*^0 \left( u_* + \frac{1}{2} \mathbf{v}_*^{in2} \right) - \mathbf{v}_+ \cdot \llbracket \mathbf{S}_+ \rrbracket \cdot \mathbf{N} - \mathbf{v}_*^{in} \cdot \llbracket \mathbf{S}_* \rrbracket \cdot \mathbf{N} + \rho_*^0 \llbracket \mathbf{v}_* \rrbracket \langle \mathbf{v}_* \rangle \mathbf{V}_*^{ex} \cdot \mathbf{N} + (1 - \alpha)q_\Gamma \geq 0. \quad (73)$$

Let  $f_{+,*} = -Ts_{+,*} + u_{+,*}$  are the densities of Helmholtz free energy, and finally we get

$$D = -(\hat{\rho}_+^0 \left( f_+ + \frac{1}{2} \mathbf{v}_+^2 \right) + \hat{\rho}_*^0 \left( f_* + \frac{1}{2} \mathbf{v}_*^{in2} \right) + (\mathbf{v}_+ \cdot \llbracket \mathbf{S}_+ \rrbracket + \mathbf{v}_*^{in} \llbracket \mathbf{S}_* \rrbracket) \cdot \mathbf{N} - \rho_*^0 \llbracket \mathbf{v}_* \rrbracket \langle \mathbf{v}_* \rangle \mathbf{V}_*^{ex} \cdot \mathbf{N} + (1 - \alpha)q_\Gamma \geq 0. \quad (74)$$

**Surface growth tensor.** To obtain surface growth tensor it should be taken into account that  $\llbracket \mathbf{v}_* \rrbracket \langle \mathbf{v}_* \rangle = \llbracket \mathbf{v}_*^2 \rrbracket / 2$  and that from the momentum equation it follows that

$$\llbracket \mathbf{S}_+ \rrbracket \cdot \mathbf{N} = -\llbracket \mathbf{S}_* \rrbracket \cdot \mathbf{N} - \hat{\rho}_+^0 \mathbf{v}_+ - \hat{\rho}_*^0 \mathbf{v}_*^{in} + \rho_*^0 \llbracket \mathbf{v}_* \rrbracket \mathbf{V}_*^{ex} \cdot \mathbf{N}. \quad (75)$$

Hence following terms in Eq. (74) can be transformed as follows

$$\begin{aligned} \mathbf{v}_+ \cdot \llbracket \mathbf{S}_+ \rrbracket \cdot \mathbf{N} + \mathbf{v}_*^{in} \cdot \llbracket \mathbf{S}_* \rrbracket \cdot \mathbf{N} - \rho_*^0 \frac{\llbracket \mathbf{v}_*^2 \rrbracket}{2} \mathbf{V}_*^{ex} \cdot \mathbf{N} &= \mathbf{v}_\Gamma^t \cdot \llbracket \mathbf{S}_+ \rrbracket \cdot \mathbf{N} + \mathbf{v}_*^{in} \cdot \llbracket \mathbf{S}_* \rrbracket \cdot \mathbf{N} + \\ &+ (\mathbf{v}_+ - \mathbf{v}_\Gamma^t) \cdot \llbracket \mathbf{S}_+ \rrbracket \cdot \mathbf{N} - \rho_*^0 \frac{\llbracket \mathbf{v}_*^2 \rrbracket}{2} \mathbf{V}_*^{ex} \cdot \mathbf{N} = \mathbf{v}_*^{in} \cdot \llbracket \mathbf{S}_* \rrbracket \cdot \mathbf{N} - \rho_*^0 \frac{\llbracket \mathbf{v}_*^2 \rrbracket}{2} \mathbf{V}_*^{ex} \cdot \mathbf{N} + \\ &+ \mathbf{v}_\Gamma^t \cdot (-\llbracket \mathbf{S}_* \rrbracket \cdot \mathbf{N} - \hat{\rho}_+^0 \mathbf{v}_+ - \hat{\rho}_*^0 \mathbf{v}_*^{in} + \rho_*^0 \llbracket \mathbf{v}_* \rrbracket \mathbf{V}_*^{ex} \cdot \mathbf{N}) - \mathbf{F}_+^T \cdot \mathbf{v}_\Gamma^0 \cdot \llbracket \mathbf{S}_+ \rrbracket \cdot \mathbf{N} = \\ &= -(\mathbf{N} \cdot \mathbf{F}_+^T \cdot \llbracket \mathbf{S}_+ \rrbracket \cdot \mathbf{N})W - \mathbf{v}_\Gamma^t \hat{\rho}_+^0 \mathbf{v}_+ - \mathbf{v}_\Gamma^t \hat{\rho}_*^0 \mathbf{v}_*^{in} + (\mathbf{v}_*^{in} - \mathbf{v}_\Gamma^t) \cdot \llbracket \mathbf{S}_* \rrbracket \cdot \mathbf{N} + \\ &+ \left( \mathbf{v}_\Gamma^t \rho_*^0 \llbracket \mathbf{v}_* \rrbracket - \rho_*^0 \frac{\llbracket \mathbf{v}_*^2 \rrbracket}{2} \right) \mathbf{V}_*^{ex} \cdot \mathbf{N}. \end{aligned} \quad (76)$$

From the following equations

$$\mathbf{v}_*^{in} - \mathbf{v}_\Gamma^t = \mathbf{v}_*^{in} - \mathbf{v}_+ - \mathbf{v}_\Gamma^0 \cdot \mathbf{F}_+^T; (\mathbf{v}_*^{in} - \mathbf{v}_\Gamma^t) \cdot \mathbf{F}_+^{-T} = \mathbf{V}_*^{in} - \mathbf{v}_\Gamma^0, \quad (77)$$

it follows that

$$(\mathbf{v}_*^{in} - \mathbf{v}_\Gamma^t) \cdot \llbracket \mathbf{S}_* \rrbracket \cdot \mathbf{N} = (\mathbf{V}_*^{in} \cdot \mathbf{N} - \mathbf{v}_\Gamma^0 \cdot \mathbf{N}) \mathbf{F}_+^T \llbracket \mathbf{S}_* \rrbracket = (\mathbf{V}_*^{in} \cdot \mathbf{N} - W) \mathbf{F}_+^T \llbracket \mathbf{S}_* \rrbracket. \quad (78)$$

From mass balance it follows that

$$-\mathbf{V}_*^{in} \cdot \mathbf{N} + W = \frac{\hat{\rho}_*^0}{\rho_*^0} - \mathbf{V}_*^{ex} \cdot \mathbf{N}, \quad (79)$$

hence it can be written that

$$(\mathbf{v}_*^{in} - \mathbf{v}_\Gamma^t) \cdot \llbracket \mathbf{S}_* \rrbracket \cdot \mathbf{N} = -\frac{\hat{\rho}_*^0}{\rho_*^0} \mathbf{F}_+^T \llbracket \mathbf{S}_* \rrbracket + \mathbf{V}_*^{ex} \cdot \mathbf{F}_+^T \llbracket \mathbf{S}_* \rrbracket \cdot \mathbf{N}. \quad (80)$$

Then dissipation inequality takes the following form

$$\begin{aligned} & - \left[ \hat{\rho}_+^0 \left( f_+ + \frac{1}{2} \mathbf{v}_+^2 \right) + \hat{\rho}_*^0 \left( f_* + \frac{1}{2} \mathbf{v}_*^{in2} \right) - (\mathbf{N} \cdot \mathbf{F}_+^T \cdot \llbracket \mathbf{S}_+ \rrbracket \cdot \mathbf{N})W + \frac{\rho_+^0}{\rho_*^0} \mathbf{F}_+^T \llbracket \mathbf{S}_* \rrbracket W - \right. \\ & \left. - \mathbf{v}_\Gamma^t \hat{\rho}_*^0 \mathbf{v}_*^{in} - \mathbf{v}_\Gamma^t \rho_+ \mathbf{v}_+ W + \mathbf{V}_*^{ex} \mathbf{F}_+^T \llbracket \mathbf{S}_* \rrbracket \cdot \mathbf{N} + (\mathbf{v}_\Gamma^t \rho_*^0 \llbracket \mathbf{v}_* \rrbracket - \rho_*^0 \frac{\llbracket \mathbf{v}_*^2 \rrbracket}{2}) \mathbf{V}_*^{ex} \cdot \mathbf{N} \right] + \\ & + (1 - \alpha)q_\Gamma \geq 0. \end{aligned} \quad (81)$$

If whole diffusive constituent transforms into the solid, i.e.  $\mathbf{V}_*^{ex} = 0$ , then dissipation can be rewritten as

$$\begin{aligned}
& -\rho_+^0 \cdot \left[ \mathbf{N} \cdot (f_+ \mathbf{I} - \frac{1}{\rho_+^0} \mathbf{F}_+^T \cdot (\mathbf{S}_+^{in} - \mathbf{S}_+^{ex})) \cdot \mathbf{N} - \mathbf{N} \cdot (f_* \mathbf{I} - \frac{1}{\rho_*^0} \mathbf{F}_+^T \cdot \mathbf{S}_*) \cdot \mathbf{N} + \right. \\
& \left. + (\frac{1}{2} \mathbf{v}_+^2 - \frac{1}{2} \mathbf{v}_*^{in2} - \mathbf{v}_\Gamma^t \mathbf{v}_+ + \mathbf{v}_\Gamma^t \mathbf{v}_*^{in}) \right] W + (1 - \alpha) q_\Gamma \geq 0.
\end{aligned} \tag{82}$$

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