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NUMERICALLY-ANALYTICALLY STUDYING FUNDAMENTAL SOLUTIONS OF 3-D DYNAMICS OF PARTIALLY SATURATED POROELASTIC BODIES

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Abstract. A mathematical model of a porous material is considered, in which an elastic skeleton and two fluid phases filling the pores are discerned. The dynamic equations are written in Laplace-type representation for unknown displacement functions of the skeleton and pore pressures of the fillers. The fundamental solutions of the defining differential equations are numerically-analytically studied. A solution in the time-domain is constructed, using the time-step method of numerically inverting Laplace transform.

Keywords: elastic diffusion, unsteady problems, Green's functions, integral transformations

1. Introduction

Analyzing the dynamic behavior of poroelastic media, such as soils and rocks, is, to a large extent, is based on using the already developed numerical methods. Thus, for instance, in dealing with problems of propagation of seismic waves, the boundary element method (BEM) is more effective, as it automatically satisfies the conditions on infinity. However, use of the BEM is limited by the necessity of constructing Green functions of defining differential equations. In works [1,2] Gatmiri and Jabbari obtained fundamental solutions for static and quasi-static problems of the theory of non-saturated soils in 2-D and 3-D formulations. Fundamental solutions for dynamic 2-D problems were presented by Maghoul in [3], and for 3-D ones by Ashayeri in [4] and by Li in [5].

The present paper, following [5], studies numerically and analytically the fundamental solutions of 3-D dynamics of partially saturated poroelastic media, commenting on some errors made in [5] when determining the fundamental solutions.

2. Governing differential equations

To describe the mechanical behavior of a partially saturated porous medium, the effective stress principle is used, which was introduced by Terzaghi. The corresponding defining relations in terms of stress were formulated, based on Terzhagi's principle:

$$\sigma_{ij} = \left(K - \frac{2}{3}G\right)\delta_{ij}u_{k,k} + G(u_{i,j} + u_{j,i}) - \delta_{ij}(S_w p^w + S_a p^a), \quad (1)$$

where α is effective stress coefficient, K and G are elastic moduli of the porous material. Also S_w, S_a, p^w, p^a stand for water and air saturation, water and air pressure, respectively.

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The components of strain tensor ε_{ij} of a solid and displacements u_i are correlated by geometric relations:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) . \tag{2}$$

The momentum balance equation for a unit element of a partially saturated poroelastic medium can be written as

$$Gu_{i,jj} + \left(K + \frac{1}{3}G\right)u_{j,ij} - \alpha(S_w p^w + S_a p^a) + F_i = \rho\ddot{u}_i + \varphi S_w \rho_w \ddot{u}_i^w + \varphi S_a \rho_a \ddot{u}_i^a , \tag{3}$$

where F_i denotes bulk body force, ρ is averaged density of the mixture, φ is porosity, ρ_w, ρ_a, ρ_s are density of water, density of air and density of solid, respectively.

To describe fluid phase transfer, a dynamic version of Darcy law is used:

$$\varphi S_f u_i^f = -\kappa_f \left(p_{,i}^f + \rho_f \ddot{u}_i + \rho_f \ddot{u}_i^f \right), \tag{4}$$

where u_i^f is displacement of the fluid phase, and it is assumed that $f = a$ for air and $f = w$ for water. Phase permeability κ_f is defined as:

$$\kappa_f = K_{rf} k / \eta_f , \tag{5}$$

where k is permeability of the poroelastic material, K_{rf} is relative phase permeability, η_f is viscosity of the filler. The authors used Corey's relations for the case of a gas-water mixture in the pores:

$$K_{rw} = S_e^{(2+3\theta)/\theta}, K_{ra} = \left(1 - S_e\right)^2 \left[1 - S_e^{(2+\theta)/\theta}\right]. \tag{6}$$

In a partially saturated porous material containing two non-mixing fillers, the interface surface between them is curved as a result of intermolecular interaction forces. The accompanying pressure difference in both of the interfacing phases is called capillary pressure p^c. Capillary pressure can be represented as a function of the saturation degree as follows:

$$p^c = p^a - p^w = p^d S_e^{-\frac{1}{\theta}}, \quad S_e = \frac{S_w - S_{rw}}{S_{ra} - S_{rw}}, \tag{7}$$

where S_e is effective water-saturation; S_{rw} is residual water-saturation; S_{ra} is residual gas-saturation; p^d is gas pressure required for driving the liquid out of the pores; $\theta \in [0.2, 3]$ is skeleton grain size distribution coefficient.

The final differential equations in Laplace domain yield

$$\mathbf{B}\begin{bmatrix} \bar{u}_i \\ \bar{p}^w \\ \bar{p}^a \end{bmatrix} = \begin{bmatrix} (a_1\nabla^2 - a_2 s^2)\delta_{ij} + a_3\partial_i\partial_j & -a_4\partial_i & -a_5\partial_i \\ -a_4 s\partial_j & -a_6 s + \dfrac{a_7}{s}\nabla^2 & -a_8 \\ -a_5 s\partial_j & -a_9 s & -a_{10}s + \dfrac{a_{11}}{s}\nabla^2 \end{bmatrix} \begin{bmatrix} \bar{u}_i \\ \bar{p}^w \\ \bar{p}^a \end{bmatrix} = \begin{bmatrix} -\bar{F}_i \\ 0 \\ 0 \end{bmatrix}, \tag{8}$$

with

$$a_1 = G , \; a_2 = \rho - \beta S_w \rho_w - \gamma S_a \rho_a , \; a_3 = K + \frac{G}{3}, \; a_4 = (\alpha - \beta) S_w , \tag{9}$$

$$a_5 = (\alpha - \gamma) S_a , \; a_6 = \varsigma S_{ww} S_w + \frac{\varphi}{K_w} S_w - S_u \varphi , \; a_7 = \frac{\beta S_w}{\rho_w} , \tag{10}$$

$$a_8 = \varsigma S_{aa} S_w + S_u \varphi\,,\ a_9 = \varsigma S_{ww} S_a + S_u \varphi\,,\ a_{10} = \varsigma S_{aa} S_a + \frac{\varphi}{K_a} S_a - S_u \varphi\,,\ a_{11} = \frac{\gamma S_a}{\rho_a}\,, \tag{11}$$

where K_w and K_a are bulk moduli of the fluid. The following abbreviations

$$\varsigma = \frac{\alpha - \varphi}{K_s}\,,\ S_{ww} = S_w - \theta(S_w - S_{rw})\,,\ S_{aa} = S_a + \theta(S_w - S_{rw})\,, \tag{12}$$

$$S_u = -\frac{\theta(S_{ra} - S_{rw})}{p^d} \left(\frac{S_w - S_{rw}}{S_{ra} - S_{rw}} \right)^{\frac{\theta+1}{\theta}} \tag{13}$$

are introduced, where S_{rw} is residual water saturation and S_{ra} is air saturation. Symbols β and γ are Laplace parameter dependent variables expressed as

$$\beta = \frac{\kappa_w \varphi \rho_w s}{\varphi S_w + \kappa_w \rho_w s}\,,\ \gamma = \frac{\kappa_a \varphi \rho_a s}{\varphi S_a + \kappa_a \rho_a s}\,. \tag{14}$$

Dynamic equations of a partially saturated poroelastic medium (8) differ from the equations in [5] in that in coefficients a_6, a_8, a_9, a_{10} they contain summand $S_u\varphi$ instead of the incorrect S_u.

3. Green functions

A fundamental solution of system (8) was found in [5], using Hormander's method, from relation.

$$\mathbf{B}^*\overline{\mathbf{U}} + \mathbf{I}\delta(\mathbf{x},\mathbf{y}) = \mathbf{0}\,, \tag{15}$$

where $\mathbf{B}^*$ is operator coupled with $\mathbf{B}$, $\mathbf{I}$ is unit matrix, $\delta(\mathbf{x},\mathbf{y})$ is Dirac delta-function, $\mathbf{x},\mathbf{y} \in R^3$.

Substituting

$$\mathbf{G} = \mathbf{B}^{*co}\phi\,, \tag{16}$$

where $\mathbf{B}^{*co}$ is algebraic complementation matrix, results in the following form of equation (15)

$$\det(\mathbf{B}^*)\phi + \delta(\mathbf{x},\mathbf{y}) = 0\,, \tag{17}$$

with unknown scalar function ϕ.

The determinant of matrix $\mathbf{B}^*$ has the following form:

$$\det(\mathbf{B}^*) = \left(a_1\nabla^2 - a_2 s^2\right)^2 \left(C_1\nabla^6 + C_2\nabla^4 + C_3\nabla^2 + C_4\right), \tag{18}$$

where:

$$C_1 = \frac{(a_1 + a_3)a_7 a_{11}}{s^2}\,, \tag{19}$$

$$C_2 = -((a_1 + a_3)(a_6 a_{11} + a_{10} a_7) + a_2 a_7 a_{11} + a_4^2 a_{11} + a_5^2 a_7)\,, \tag{20}$$

$$C_3 = s^2((a_6 a_{10} - a_8 a_9)(a_1 + a_3) + a_2(a_6 a_{11} + a_7 a_{10}) + a_4^2 a_{10} + a_5^2 a_6 - a_4 a_5 (a_8 + a_9))\,, \tag{21}$$

$$C_4 = s^4 a_2 (a_8 a_9 - a_6 a_{10})\,. \tag{22}$$

The expression for the determinant of matrix $\mathbf{B}^*$ from [5] contains the wrong sign at summand $a_4 a_5 (a_8 + a_9)$ in coefficient C_3.

Factorization of expression (18) using Kardano's formula makes it possible to rewrite equation (17) for unknown function ϕ as:

$$\left(\nabla^2 - \lambda_1^2\right)\left(\nabla^2 - \lambda_2^2\right)\left(\nabla^2 - \lambda_3^2\right)\left(\nabla^2 - \lambda_4^2\right)^2 \phi = 0\,, \tag{23}$$

where

$$\lambda_1 = \sqrt{N_1 + \frac{N_2 C_2^2}{3C_1} - N_2 C_3 + \frac{1}{3N_2 C_1}}, \tag{24}$$

$$\lambda_2 = \sqrt{N_1 + \frac{3C_1C_3 - C_2^2}{3C_1} N_2 \left(1 - i\sqrt{3}\right) - \frac{1}{6N_2C_1}\left(1 + i\sqrt{3}\right)}, \tag{25}$$

$$\lambda_3 = \sqrt{N_1 + \frac{3C_1C_3 - C_2^2}{3C_1} N_2 \left(1 + i\sqrt{3}\right) - \frac{1}{6N_2C_1}\left(1 - i\sqrt{3}\right)}, \tag{26}$$

$$\lambda_4 = \sqrt{\frac{a_2}{a_1} s}, \tag{27}$$

$$N_1 = -\frac{C_2}{3C_1},\ N_2 = \frac{\sqrt[3]{2}}{N3}, \tag{28}$$

$$N_3 = \sqrt[3]{-2C_2^3 + 9C_1C_2C_3 - 27C_1^2C_4 + \sqrt{4\left(-C_2^2 + 3C_1C_3\right)^3 + \left(-2C_2^3 + 9C_1C_2C_3 - 27C_1^2C_4\right)^2}}\ . \tag{29}$$

The form of function ϕ, as well as the expressions for the components of the fundamental solutions, are presented in [5] and are used in the further calculations, taking account of the above mentioned comments.

4. Numerical results

The results are exemplified by the visualization of the components of fundamental solutions U_{11}^{ss}, P^{ww}, and P^{aa}. The values in the time-domain are obtained using the time-step method of numerically inverting Laplace transform [6,7] with parameters $\Delta t = 10^{-6}$, L=1000, N= 1000, R=0.997. The parameters of the partially saturated porous material correspond to those of sandstone: $\varphi = 0.23$, $\rho_s = 2650\, kg/m^3$, $\rho_w = 1.0\, kg/m^3$, $\rho_w = 997\, kg/m^3$, $\rho_a = 1.1 kg/m^3$, $K = 1.02 \cdot 10^9\, N/m^2$, $G = 1.44 \cdot 10^9\, N/m^2$, $K_s = 3.5 \cdot 10^{10}\, N/m^2$, $K_w = 2.25 \cdot 10^9\, N/m^2$, $K_a = 1.1 \cdot 10^5\, N/m^2$, $k = 2.5 \cdot 10^{-12}\, m^2$, $\eta_w = 1.0 \cdot 10^{-3}\, N \cdot s/m^2$, $\eta_a = 1.8 \cdot 10^{-5}\, N \cdot s/m^2$, $p^d = 5 \cdot 10^4\, N/m^2$, $S_w = 0.95$, $S_{rw} = 0$, $S_{ra} = 1$, $\theta = 1.5$. An observation point with coordinates $(0.5, 0, 0)$ and a load application point with coordinates $(0, 0, 0)$ are chosen.

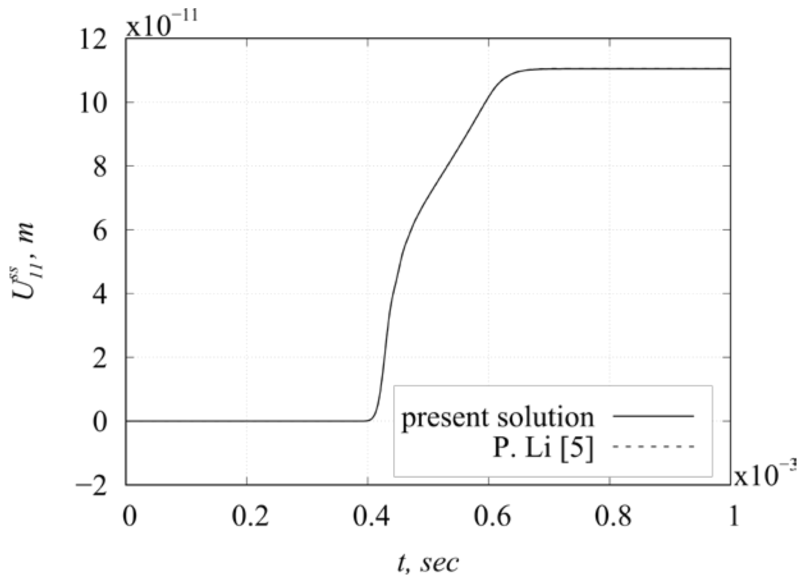

Fig. 1. Displacement U_{11}^{ss} versus time

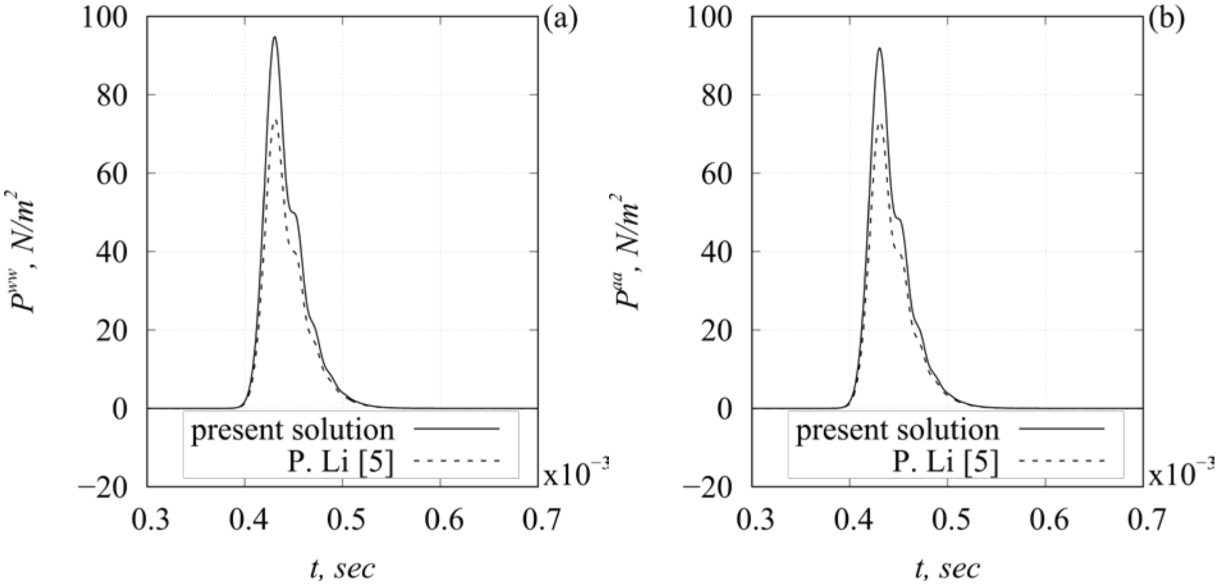

Fig. 2. Pore water pressure (a) and pore air pressure (b)

It can be observed in Fig. 1 that the values of U_{11}^{ss} computed using expressions (10), (11), (22) do not differ graphically from those obtained using the expressions from [5]. However, the results of computing components P^{ww} and P^{aa} demonstrate significant difference. In particular, the expressions used in [5] give underestimated values of P^{ww} and P^{aa} at the time the fast longitudinal wave arrives (Fig. 2).

5. Conclusion

A visualization of the components of the fundamental solutions of dynamic poroelasticity is presented. It is indicated that the solutions presented in [5] contain errors. The effect of the above errors on the results of numerical experiments is demonstrated.

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References

[1] Gatmiri B, Jabbari E. Time-domain Green's functions for unsaturated soils. Part I: Two-dimensional solution. *International Journal of Solids and Structures.* 2005;42(23): 5971-5990.

[2] Gatmiri B, Jabbari E. Time-domain Green's functions for unsaturated soils. Part II: Three-dimensional solution. *International Journal of Solids and Structures.* 2005;42(23): 5991-6002.

[3] Maghoul P, Gatmiri B, Duhamel D. Boundary integral formulation and two-dimensional fundamental solutions for dynamic behavior analysis of unsaturated soils. *Soil Dynamics and Earthquake Engineering.* 2011;31(11): 1480-1495.

[4] Ashayeri I, Kamalian M, Jafari MK, Gatmiri B. Analytical 3D transient elastodynamic fundamental solution of unsaturated soils. *International Journal for Numerical and Analytical Methods in Geomechanics.* 2011;35(17): 1801-1829.

[5] Li P. *Boundary element method for wave propagation in partially saturated poroelastic continua*. Graz: Verlag der Technischen Universität Graz; 2012.

[6] Igumnov LA, Petrov AN. Dynamics of partially saturated poroelastic solids by boundary-element method. *PNRPU Mechanics Bulletin.* 2016;3: 47-61. (In Russian)

[7] Belov AA, Igumnov LA, Petrov AN. The stepped analysis of the effect of the saturation coefficient on dynamic pore responses. *Problems of Strength and Plasticity*. 2017;79(4): 425-436. (In Russian)