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REFLECTION AND TRANSMISSION OF PLANE WAVES AT THE INTERFACE OF AN ELASTIC HALF-SPACE AND A MICROPOLAR THERMOELASTIC HALF-SPACE WITH FRACTIONAL ORDER DERIVATIVE

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Abstract. The problem of reflection and transmission due to longitudinal and transverse waves incident obliquely at a plane interface between uniform elastic solid half-space and micropolar thermoelastic solid half-space with fractional order derivative has been studied. It is found that the amplitude ratios of various reflected and transmitted waves are functions of angle of incidence, frequency of incident wave and are influenced by the fractional order and micropolar thermoelastic properties of media. The expressions of amplitude ratios and energy ratios have been computed numerically for a particular model. The variations of energy ratios for different fractional orders with angle of incidence for Lord-Shulman [1] and Green-Lindsay [2] theories of thermoelasticity are shown graphically for insulated boundary. The conservation of energy at the interface is verified.

1. Introduction

The generalized theory of thermoelasticity is one of the modified versions of classical uncoupled and coupled theory of thermoelasticity and has been developed in order to remove the paradox of physical impossible phenomena of infinite velocity of thermal signals in the classical coupled thermoelasticity.

Several generalizations to the coupled theory are introduced. One can refer to Ignaczak [3] and to Chandrasekharaiah [4] for a review, presentation of generalized theories. Hetnarski and Ignaczak in their survey article [5] examined five generalizations to the coupled theory and obtained a number of important analytical results. The mathematical aspects of Lord-Shulman [1] and Green-Lindsay [2] theories and convolutional variational principles are explained and illustrated in the work of Ignaczak and Ostoja-Starzeweski [6].

The linear theory of micropolar thermoelasticity was developed by extending the theory of micropolar continua to include thermal effects by Eringen [7] and Nowacki [8]. Different authors [9-12] discussed different problems in micropolar elasticity/micropolar theory of thermoelasticity.

During recent years, several interesting models have been developed by using fractional calculus to study the physical processes particularly in the area of heat conduction, diffusion, viscoelasticity, mechanics of solids, control theory, electricity etc. It has been realized that the use of fractional order derivatives and integrals leads to the formulation of certain physical problems which is more economical and useful than the classical approach.

The first application of fractional derivatives was given by Abel [13] who applied fractional calculus in the solution of an integral equation that arises in the formulation of the tautochrone

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problem. Caputo [14] gave the definition of fractional derivatives of order $\alpha \in (0,1]$ of absolutely continuous function. Caputo and Mainardi [15], Caputo and Mainardi [16] and Caputo [17] found good agreement with experimental results when using fractional derivatives for description of viscoelastic materials and established the connection between fractional derivatives and the theory of linear viscoelasticity.

Oldham and Spanier [18] studied the fractional calculus and proved the generalization of the concept of derivative and integral to a non-integer order. A theoretical basis for the application of fractional calculus to viscoelasticity was given by Bagley and Torvik [19]. Applications of fractional calculus to the theory of viscoelasticity were given by Koeller [20]. Kochubei [21] studied the problem of fractional order diffusion. Rossikhin and Shitikova [22] presented applications of fractional calculus to various problems of mechanics of solids. Gorenflo and Mainardi [23] discussed the integral differential equations of fractional orders, fractals and fractional calculus in continuum mechanics.

Povstenko [24] investigated the nonlocal generalizations of the Fourier law and heat conduction by using time and space fractional derivatives. Jiang and Xu [25] obtained a fractional heat conduction equation with a time fractional derivative in the general orthogonal curvilinear coordinate and also in other orthogonal coordinate system. Povstenko [26] investigated the fractional radial heat conduction in an infinite medium with a cylindrical cavity and associated thermal stresses.

Ezzat [27] constructed a new model of the magneto-thermoelasticity theory in the context of a new consideration of heat conduction with fractional derivative. Ezzat [28] studied the problem of state space approach to thermoelectric fluid with fractional order heat transfer. The Laplace transform and state-space techniques were used to solve a one-dimensional application for a conducting half space of thermoelectric elastic material. Povstenko [29] investigated the generalized Cattaneo-type equations with time fractional derivatives and formulated the theory of thermal stresses. Biswas and Sen [30] proposed a scheme for optimal control and a pseudo state space representation for a particular type of fractional dynamical equation. Various authors [31-34] studied different problems of reflection and transmission in different types of media.

In the present paper, the reflection and transmission phenomenon at a plane interface between an elastic solid half-space and a micropolar thermoelastic solid half-space with fractional order derivative has been analyzed. The amplitude ratios of various reflected and transmitted waves to that of incident wave are derived. The amplitude ratios are further used to find the expressions of energy ratios of various reflected and transmitted waves to that of incident wave. The effect of relaxation times and fractional orders are shown on the energy ratio graphically for a specific model for insulated boundary. The law of conservation of energy at the interface is verified.

2. Governing equations

Following Kumar and Deswal [35], Ezzat [36], the field equations and stress-strain temperature relations in micropolar generalized thermoelastic solid with fractional order derivative without body forces, body couples and heat sources can be written as

$$(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})+(\mu+K)\nabla^2\mathbf{u}+K\nabla\times\boldsymbol{\phi}-\nu\left(1+\frac{(\tau_1)^{\alpha}}{\alpha!}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\nabla T=\rho\frac{\partial^2\mathbf{u}}{\partial t^2}, \tag{1}$$

$$\gamma\nabla(\nabla\cdot\boldsymbol{\phi})+K\nabla\times\mathbf{u}-2K\boldsymbol{\phi}=\rho j\frac{\partial^2\boldsymbol{\phi}}{\partial t^2}, \tag{2}$$

$$K^*\nabla^2 T = \rho C^*\left(\frac{\partial}{\partial t} + \frac{(\tau_0)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)T + \nu T_0\left(\frac{\partial}{\partial t} + m_0\frac{(\tau_0)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)\nabla.\mathbf{u}, \tag{3}$$

$$t_{ij} = \lambda u_{r,r}\delta_{ij} + \mu\left(u_{i,j} + u_{j,i}\right) + K\left(u_{j,i} - \varepsilon_{ijr}\phi_r\right) - \nu\left(T + \frac{(\tau_1)^\alpha}{\alpha!}\frac{\partial^\alpha T}{\partial t^\alpha}\right)\delta_{ij}, \tag{4}$$

$$m_{ij} = \alpha\phi_{r,r}\delta_{ij} + \beta\phi_{i,j} + \gamma\phi_{j,i}. \tag{5}$$

The strain - displacement relations

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{6}$$

where λ, μ, κ, β, γ are material constants, ρ is density, j is the microinertia, K^* is the coefficient of thermal conductivity, $\nu = (3\lambda + 2\mu + K)\alpha_t$, C^* is the specific heat at constant strain, α_t is the coefficient of thermal linear expansion, α denotes the fractional order, $\mathbf{u}$ is the displacement vector, T is the change in temperature of the medium at any time, T_0 is the reference temperature of the body, τ_0, τ_1 are the thermal relaxation times, ϕ is the microrotation vector. For the Lord-Shulman (L-S) theory τ_1=0, m_0=1 and for Green-Lindsay (G-L) theory τ_1>0 and m_0=0. The thermal relaxations τ_0 and τ_1 satisfy the inequality $\tau_1 \geq \tau_0 \geq 0$ for the G-L theory only.

3. Formulation of the problem

We consider an isotropic elastic solid half-space (Medium I) lying over a homogeneous isotropic, micropolar thermoelastic half-space with fractional order derivative (Medium II). The origin of the Cartesian coordinate system (x_1, x_2, x_3) is taken at any point on the plane surface (interface) and x_3-axis points vertically downwards into the micropolar thermoelastic half-space with fractional order derivative which is thus represented by $x_3 \geq 0$. We choose x_1-axis in the direction of wave propagation so that all particles on a line parallel to x_2-axis are equally displaced. Therefore all the field quantities are independent of x_2.

For two dimensional problem, we take

$$\mathbf{u} = (u_1, 0, u_3), \quad \phi = (0, \phi_2, 0). \tag{7}$$

We define the following dimensionless quantities

$$x_i' = \frac{\omega^*}{c_1}x_i, \quad u_i' = \frac{\rho\omega^* c_1}{\nu T_0}u_i, \quad u_i'^e = \frac{\rho\omega^* c_1}{\nu T_0}u_i^e, \; i = 1,2,3, \quad \Phi_2' = \frac{\rho c_1^2}{\nu T_0}\Phi_2, \quad m_{ij}' = \frac{\omega^*}{c_1\nu T_0}m_{ij},$$

$$t' = \omega^* t, \; T' = \frac{T}{T_0}, \; \tau_1' = \omega^*\tau_1, \; \tau_0' = \omega^*\tau_0, \; t_{ij}' = \frac{t_{ij}}{\nu T_0}, \; t_{ij}'^e = \frac{t_{ij}^e}{\nu T_0}, \; P^{*e'} = \frac{\rho c_1}{\nu^2 T_0^2}P^{*e}, \; P_{ij}^{*'} = \frac{\rho c_1}{\nu^2 T_0^2}P_{ij}^*, \tag{8}$$

where

$$\omega^* = \frac{\rho C^* c_1^2}{K^*}, \quad c_1^2 = \frac{\lambda + 2\mu + K}{\rho}. \tag{9}$$

Equations (1)-(3) with the aid of (7) and (8) after suppressing the primes reduce to

$$\frac{(\lambda+\mu)}{\rho c_1^2}\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_3}{\partial x_3}\right)+\frac{(\mu+K)}{\rho c_1^2}\nabla^2 u_1-\frac{K}{\rho c_1^2}\frac{\partial \phi_2}{\partial x_3}-\left(1+\frac{(\tau_1)^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial T}{\partial x_1}=\frac{\partial^2 u_1}{\partial t^2}, \tag{10}$$

$$\frac{(\lambda+\mu)}{\rho c_1^2}\frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_3}{\partial x_3}\right)+\frac{(\mu+K)}{\rho c_1^2}\nabla^2 u_3+\frac{K}{\rho c_1^2}\frac{\partial \phi_2}{\partial x_3}-\left(1+\frac{(\tau_1)^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial T}{\partial x_3}=\frac{\partial^2 u_3}{\partial t^2}, \tag{11}$$

$$\nabla^2\phi_2+\frac{Kc_1}{\gamma\omega^{*2}}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_3}{\partial x_3}\right)-\frac{2Kc_1^2}{\gamma\omega^{*2}}\phi_2=\frac{\rho j c_1^2}{\gamma}\frac{\partial^2\phi_2}{\partial t^2}, \tag{12}$$

$$\nabla^2 T-\left(\frac{\partial T}{\partial t}+\frac{(\tau_0)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}T}{\partial t^{\alpha+1}}\right)=\frac{\nu^2 T_0}{\rho K^*\omega^*}\left(\frac{\partial}{\partial t}+m_0\frac{(\tau_0)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_3}{\partial x_3}\right). \tag{13}$$

We introduce the potential functions Φ and Ψ through the relations

$$u_1=\frac{\partial\Phi}{\partial x_1}-\frac{\partial\Psi}{\partial x_3}, \quad u_3=\frac{\partial\Phi}{\partial x_3}+\frac{\partial\Psi}{\partial x_1}. \tag{14}$$

Substituting equation (14) in the equations (10)-(13), we obtain

$$\nabla^2\Phi-\tau_{11}T-\frac{\partial^2\Phi}{\partial t^2}=0, \tag{15}$$

$$a_1\nabla^2\Psi-\frac{\partial^2\Psi}{\partial t^2}+a_2\phi_2=0, \tag{16}$$

$$\left(\nabla^2-2a_3-a_4\frac{\partial^2}{\partial t^2}\right)\phi_2-a_3\nabla^2\Psi=0, \tag{17}$$

$$\nabla^2 T-\left(\frac{\partial T}{\partial t}+\frac{(\tau_0)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}T}{\partial t^{\alpha+1}}\right)+a_5\left(\frac{\partial}{\partial t}+m_0\frac{(\tau_0)^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)\nabla^2\Phi=0, \tag{18}$$

where

$$\tau_{11}=\left(1+\frac{(\tau_1)^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right), \quad a_1=\frac{(\mu+K)}{\rho c_1^2}, \quad a_2=\frac{K}{\rho c_1^2}, \quad a_3=\frac{Kc_1^2}{\gamma\omega^{*2}}, \quad a_4=\frac{\rho j c_1^2}{\gamma}, \quad a_5=\frac{\nu^2 T_0}{\rho K^*\omega^*}.$$

For the propagation of harmonic waves in x_1x_3 - plane, we assume

$$\{\Phi, \Psi, \phi_2, T\}(x_1, x_3, t) = \{\bar{\Phi}, \bar{\Psi}, \bar{\phi}_2, \bar{T}\} e^{-i\omega t}, \tag{19}$$

where ω is the angular frequency of vibrations of material particles.
Substituting the values of Φ, Ψ, ϕ_2 and T from equation (19) in equations (15)-(18) after simplification, we obtain

$$\left[A\nabla^4 + B\nabla^2 + C\right]\bar{\Psi} = 0, \tag{20}$$

$$\left[D\nabla^4 + E\nabla^2 + F\right]\bar{\Phi} = 0, \tag{21}$$

where
the general solution of equation (21) can be written as

$$\bar{\Phi} = \bar{\Phi}_1 + \bar{\Phi}_2, \tag{22}$$

where the potentials $\bar{\Phi}_i$, i=1, 2 are solutions of wave equations, given by

$$\left[\nabla^2 + \frac{\omega^2}{V_i^2}\right]\bar{\Phi}_i = 0 \text{ , i=1, 2.} \tag{23}$$

Here V_1, V_2 are the velocities of coupled longitudinal waves, that is, longitudinal displacement wave (LD) and Thermal wave (T) and are derived from the roots of quadratic equations in V^2, given by

$$FV^4 - E\omega^2V^2 + D\omega^4 = 0. \tag{24}$$

Using equation (18), (23) with the aid of (19) and (22), we obtain

$$\{\Phi, T\} = \sum_{i=1}^{2}\{1, n_i\}\Phi_i, \tag{25}$$

where

$$n_i = \frac{\beta^2\omega^4 C\bar{\Phi}}{\left(\omega^2 CV_i^2 - \omega^2\left(C_T^2 A - i\omega BC_K^2\right)\right)}, \text{ i=1, 2.} \tag{26}$$

The general solution of equation (20) can be written as

$$\bar{\Psi} = \bar{\Psi}_1 + \bar{\Psi}_2, \tag{27}$$

where the potentials $\bar{\Psi}_i$, i=3, 4 are solutions of wave equations, given by

$$\left[\nabla^2 + \frac{\omega^2}{V_i^2}\right]\bar{\Psi}_i = 0, \text{ i=3,4.} \tag{28}$$

Here V_3, V_4 are the velocities of two transverse displacement waves coupled with microrotation wave that is, (CDI, CDII) and derived from the roots of quadratic equations in V^2, given by

$$CV^4 - B\omega^2V^2 + A\omega^4 = 0. \tag{29}$$

Using equation (17), (28) with the aid of (19) and (27), we obtain

$$\{\Psi, \phi_2\} = \sum_{i=1}^{2} \{1, n_i\}\Psi_i, \tag{30}$$

where

$$n_i = \frac{-\omega^2 a_3}{V_i^2\left(\nabla^2 - 2a_3 + \omega^2 a_4\right)}, \quad i = 3, 4. \tag{31}$$

The basic equations in a homogeneous isotropic elastic solid are

$$\left(\lambda^e + \mu^e\right)\nabla\left(\nabla.\mathbf{u}^e\right) + \mu\nabla^2\mathbf{u}^e = \rho^e \frac{\partial^2 \mathbf{u}^e}{\partial t^2}, \tag{32}$$

where λ^e, μ^e are the Lame's constants, $\mathbf{u}^e$ is the displacement vector, ρ^e is the density corresponding to medium I.

For two dimensional problem, the components of $\mathbf{u}^e = \left(u_1^e, 0, u_3^e\right)$ can be written as

$$u_1^e = \frac{\partial \Phi^e}{\partial x_1} - \frac{\partial \Psi^e}{\partial x_3}, \quad u_3^e = \frac{\partial \Phi^e}{\partial x_3} + \frac{\partial \Psi^e}{\partial x_1}, \tag{33}$$

where Φ^e and Ψ^e are the potential functions satisfying the wave equations

$$\nabla^2\Phi^e = \frac{1}{\alpha^2}\ddot{\Phi}^e, \tag{34}$$

$$\nabla^2\Psi^e = \frac{1}{\beta^2}\ddot{\Psi}^e, \tag{35}$$

where

$$\alpha = \frac{\alpha^e}{c_1}, \quad \beta = \frac{\beta^e}{c_1}, \quad \alpha^e = \sqrt{\frac{\lambda^e + 2\mu^e}{\rho^e}}, \quad \beta^e = \sqrt{\frac{\mu^e}{\rho^e}}. \tag{36}$$

The stress tensor σ_{ij}^e and strain tensor e_{ij}^e in the isotropic elastic medium are given by

$$\sigma_{ij}^e = 2\mu^e e_{ij}^e + \lambda^e e_{kk}^e \delta_{ij},$$

$$e_{ij}^{\ e} = \frac{1}{2}\left(u_{i,j}^{e} + u_{j,i}^{e}\right). \tag{37}$$

where $e_{ii}^{\ e}$ is the dilatation.

4. Reflection and transmission

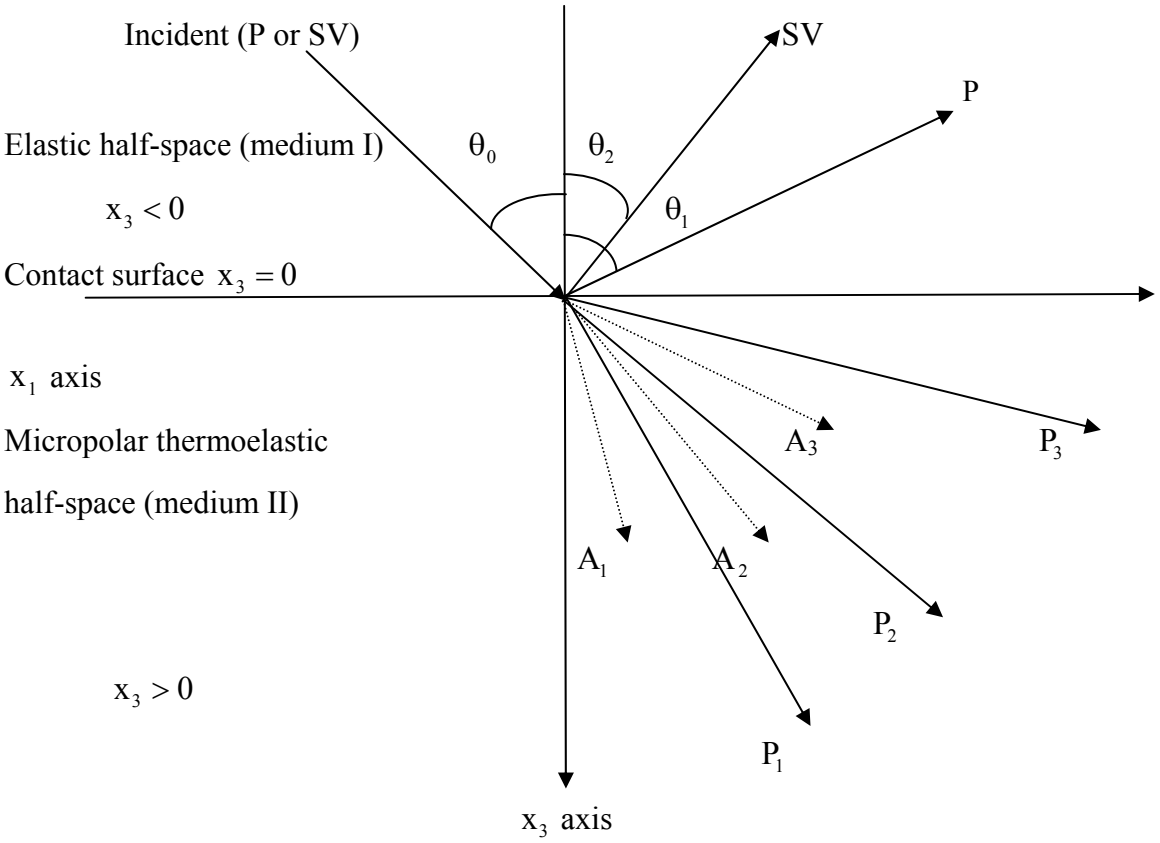

Fig. 1. Geometry of the problem.

We consider a harmonic wave (P or SV) propagating through the isotropic elastic solid half-space and is incident at the interface $x_3 = 0$. Corresponding to this incident wave, two homogeneous waves (P and SV) are reflected in isotropic elastic solid and four inhomogeneous waves (LD, T, CDI and CDII) are transmitted in isotropic micropolar thermoelastic solid half-space with fractional order derivative.

In elastic solid half-space, the potential functions satisfying equations (34) and (35) can be written as

$$\Phi^{e} = A_0^{e} e^{\left[i\omega\{(x_1 \sin\theta_0 + x_3 \cos\theta_0)/\alpha - t\}\right]} + A_1^{e} e^{\left[i\omega\{(x_1 \sin\theta_1 + x_3 \cos\theta_1)/\alpha - t\}\right]}, \tag{38}$$

$$\Psi^{e} = B_0^{e} e^{\left[i\omega\{(x_1 \sin\theta_0 + x_3 \cos\theta_0)/\beta - t\}\right]} + B_1^{e} e^{\left[i\omega\{(x_1 \sin\theta_2 + x_3 \cos\theta_2)/\beta - t\}\right]}. \tag{39}$$

The coefficients $A_0^{e}\left(B_0^{e}\right)$, A_1^{e} and B_1^{e} are amplitudes of the incident P (or SV), reflected P and reflected SV waves respectively.

Following Borcherdt [37], in isotropic micropolar thermoelastic half-space with fractional order derivative, the potential functions satisfying equations (23) and (28) can be written as

$$\{\Psi, T\} = \sum_{i=1}^{2} \{1, n_i\} B_i e^{(\vec{A}_i \cdot \vec{r})} e^{\{i(\vec{P}_i \cdot \vec{r} - \omega t)\}}, \tag{40}$$

$$\{\Psi, \phi_2\} = \sum_{i=3}^{4} \{1, n_i\} B_i e^{(\vec{A}_i \cdot \vec{r})} e^{\{i(\vec{P}_i \cdot \vec{r} - \omega t)\}}. \tag{41}$$

The coefficients B_i, $i = 1, 2, 3, 4$ are the amplitudes of transmitted P, T and SV waves respectively. The propagation vector $\mathbf{P}_i$, $i = 1, 2, 3, 4$ and attenuation factor $\mathbf{A}_i$, $i = 1, 2, 3, 4$ are given by

$$\mathbf{P}_i = \xi_R \hat{x}_1 + dV_{iR} \hat{x}_3, \mathbf{A}_i = -\xi_I \hat{x}_1 - dV_{iI} \hat{x}_3, \quad i = 1, 2, 3, 4, \tag{42}$$

$$dV_i = dV_{iR} + i dV_{iI} = \text{p.v.} \left(\frac{\omega^2}{V_i^2} - \xi^2 \right), \quad i = 1, 2, 3, 4, \tag{43}$$

and $\xi = \xi_R + i\xi_I$ is the complex wave number. The subscripts R and I denote the real and imaginary parts of the corresponding complex number and p.v. stands for the principal value of the complex quantity derived from square root. Condition $\xi_R \geq 0$ ensures propagation in positive x_1-direction. The complex wave number ξ in the isotropic fractional order thermoelastic medium is given by

$$\xi = \left|\vec{P}_i\right| \sin \theta_i' - i \left|\vec{A}_i\right| \sin\left(\theta_i' - \gamma_i\right), \; i = 1, 2, 3, \tag{44}$$

where γ_i, i =1, 2, 3, 4 is the angle between the propagation and attenuation vector and θ_i', i =1, 2, 3, 4 are the angles of transmission in medium II.

5. Boundary conditions

The boundary conditions are the continuity of stress and displacement components, vanishing of the tangential couple stress and gradient of temperature or temperature distribution. Mathematically these can be written as

a) Continuity of stress components

$$t_{33}^{\,e} = t_{33}, \tag{45}$$

$$t_{31}^{\,e} = t_{31}. \tag{46}$$

b) Continuity of displacement components

$$u_1^{\,e} = u_1, \tag{47}$$

$$u_3^{\,e} = u_3, \tag{48}$$

c) Vanishing of tangential couple stress components

$$m_{32} = 0, \tag{49}$$

d) Thermal boundary

$$\frac{\partial T}{\partial x_3} + hT = 0, \tag{50}$$

where h is the heat transfer coefficient, $h \to 0$ corresponds to insulated boundary, $h \to \infty$ corresponds to isothermal boundary.
Making use of potentials given by equations (38)-(41), we find that the boundary conditions are satisfied if and only if

$$\xi_R = \frac{\omega \sin\theta_0}{V_0} = \frac{\omega \sin\theta_1}{\alpha} = \frac{\omega \sin\theta_2}{\beta}, \tag{51}$$

where

$$V_0 = \begin{cases} \alpha, \text{ for incident P-wave} \\ \beta, \text{ for incident SV-wave} \end{cases} \tag{52}$$

and $\xi_l = 0$.

It means that waves are attenuating only in x_3-direction. From equation (44), it implies that if $|\vec{A}_i| \neq 0$, then $\gamma_i = \theta_i'$, i=1, 2, 3, 4, that is, attenuated vectors for the four transmitted waves are directed along the x_3 - axis.

Using equations (38)-(41) in the boundary conditions (45)-(50) with the aid of a equations (14), (33), (51), (52), we get a system of six non-homogeneous equations which can be written as

$$\sum_{j=1}^{6} d_{ij} Z_j = g_i, \tag{53}$$

where Z_j, j =1, 2, 3, 4, 5, 6 are the ratios of amplitudes of reflected P-, reflected SV-, transmitted LD-, transmitted T- , transmitted CDI- and transmitted CDII-wave to that of amplitude of incident wave;

$$d_{11} = 2\mu^e \left(\frac{\xi_R}{\omega}\right)^2 - \rho^e c_1^{\,2}, \quad d_{12} = 2\mu^e \frac{\xi_R}{\omega}\frac{dV_\beta}{\omega},$$

$$d_{13} = \lambda\left(\frac{\xi_R}{\omega}\right)^2 + \rho c_1^{\,2}\left(\frac{dV_1}{\omega}\right)^2 + \frac{\rho c_1^{\,2} n_1}{\omega^2}\left[1 + \frac{(\tau_1)^\alpha}{\alpha!}(-i\omega)^\alpha\right],$$

$$d_{14} = \lambda\left(\frac{\xi_R}{\omega}\right)^2 + \rho c_1^{\,2}\left(\frac{dV_2}{\omega}\right)^2 + \frac{\rho c_1^{\,2} n_2}{\omega^2}\left[1 + \frac{(\tau_1)^\alpha}{\alpha!}(-i\omega)^\alpha\right],$$

$$d_{15} = (2\mu + K)\frac{\xi_R}{\omega}\frac{dV_3}{\omega}, \quad d_{16} = (2\mu + K)\frac{\xi_R}{\omega}\frac{dV_4}{\omega}, \quad d_{21} = 2\mu^e \frac{\xi_R}{\omega}\frac{dV_\alpha}{\omega},$$

$$d_{22} = \mu^e\left[\left(\frac{dV_\beta}{\omega}\right)^2 - \left(\frac{\xi_R}{\omega}\right)^2\right], \quad d_{23} = (2\mu + K)\frac{\xi_R}{\omega}\frac{dV_1}{\omega}, \quad d_{24} = 2\mu\frac{\xi_R}{\omega}\frac{dV_1}{\omega} + K\frac{\xi_R}{\omega}\frac{dV_2}{\omega},$$

$$d_{25} = \mu\left[\left(\frac{\xi_R}{\omega}\right)^2 - \left(\frac{dV_3}{\omega}\right)^2\right] + K\left[\frac{n_3}{\omega^2} - \left(\frac{dV_3}{\omega}\right)^2\right],$$

$$d_{26} = \mu\left[\left(\frac{\xi_R}{\omega}\right)^2 - \left(\frac{dV_4}{\omega}\right)^2\right] + K\left[\frac{n_4}{\omega^2} - \left(\frac{dV_4}{\omega}\right)^2\right],$$

$$d_{31} = \frac{\xi_R}{\omega}, \quad d_{32} = \frac{dV_\beta}{\omega}, \quad d_{33} = -\frac{\xi_R}{\omega}, \quad d_{34} = -\frac{\xi_R}{\omega}, \quad d_{35} = \frac{dV_3}{\omega}, \quad d_{36} = \frac{dV_4}{\omega},$$

$$d_{41} = -\frac{dV_\alpha}{\omega}, \quad d_{42} = \frac{\xi_R}{\omega}, \quad d_{43} = -\frac{dV_1}{\omega}, \quad d_{44} = -\frac{dV_2}{\omega}, \quad d_{45} = -\frac{\xi_R}{\omega},$$

$$d_{46} = -\frac{\xi_R}{\omega}, \quad d_{51} = d_{52} = d_{53} = d_{54} = 0,$$

$$d_{55} = in_3\frac{dV_3}{\omega}, \quad d_{56} = in_4\frac{dV_4}{\omega}, \quad d_{61} = d_{62} = d_{65} = d_{66} = 0, \quad d_{63} = in_1\frac{dV_1}{\omega^2} + n_1\frac{h}{\omega^2},$$

$$d_{64} = in_2\frac{dV_2}{\omega^2} + n_2\frac{h}{\omega^2}, \quad \frac{dV_\alpha}{\omega} = \left(\frac{1}{\alpha^2} - \left(\frac{\xi_R}{\omega}\right)^2\right)^{\frac{1}{2}} = \left(\frac{1}{\alpha^2} - \frac{\sin^2\theta_0}{V_0^{\,2}}\right)^{\frac{1}{2}},$$

$$\frac{dV_\beta}{\omega} = \left(\frac{1}{\beta^2} - \frac{\sin^2\theta_0}{V_0^{\,2}}\right)^{\frac{1}{2}},$$

and

$$\frac{dV_j}{\omega} = \text{p.v.}\left(\frac{1}{V_j^2} - \frac{\sin^2\theta_0}{V_0^{\,2}}\right)^{\frac{1}{2}}, \quad j = 1, 2, 3, 4.$$

Here p.v. is calculated with restriction $dV_{jl} \geq 0$ to satisfy decay condition in fractional order thermoelastic medium. The coefficients g_i, i =1, 2, 3, 4, 5, 6 on the right side of the equation (53) are given by

(a) for incident P-wave

$$g_1 = -d_{11}, \quad g_2 = d_{21}, \quad g_3 = -d_{31}, \quad g_4 = d_{41}, \quad g_5 = 0, \quad g_6 = 0, \tag{54}$$

(b) for incident SV-wave

$$g_1 = d_{12}, \quad g_2 = -d_{22}, \quad g_3 = d_{32}, \quad g_4 = -d_{42}, \quad g_5 = 0, \quad g_6 = 0. \tag{55}$$

Now we consider a surface element of unit area at the interface between two media. The reason for this consideration is to calculate the partition of energy of the incident wave among the reflected and transmitted waves on both sides of the surface. Following Achenbach [38], the energy flux across the surface element, that is, rate at which the energy is communicated per unit area of the surface is represented as

$$P^* = \sigma_{lm} l_m \dot{u}_l, \tag{56}$$

where σ_{lm} is the stress tensor, l_m are the direction cosines of the unit normal $\hat{l}$ outward to the surface element and $\dot{u}_l$ are the components of the particle velocity.

The time average of P^* over a period, denoted by $\langle P^* \rangle$, represents the average energy transmission per unit surface area per unit time. Thus, on the surface with normal along x_3-direction, the average energy intensities of the waves in the elastic solid are given by

$$\langle P^{*e} \rangle = \mathrm{Re}\langle \sigma \rangle^e_{13} . \mathrm{Re}\langle \dot{u}^e_1 \rangle + \mathrm{Re}\langle \sigma \rangle^e_{33} . \mathrm{Re}\langle \dot{u}^e_3 \rangle. \tag{57}$$

Following Achenbach [38], for any two complex functions f and g, we have

$$\langle \mathrm{Re}(f) . \mathrm{Re}(g) \rangle = \frac{1}{2} \mathrm{Re}(f . \bar{g}). \tag{58}$$

The expressions for energy ratios E_i, i = 1, 2, 3, 4 for the reflected P- and reflected SV- are given by

$$E_i = -\frac{\langle P^{*e}_i \rangle}{\langle P^{*e}_0 \rangle}, \quad i = 1, 2, \tag{59}$$

where

$$\langle P^{*e}_1 \rangle = \frac{1}{2} \frac{\omega^4 \rho^e c_0^2}{\alpha} \left| A^e_1 \right|^2 \mathrm{Re}(\cos\theta_1), \quad \langle P^{*e}_2 \rangle = \frac{1}{2} \frac{\omega^4 \rho^e c_0^2}{\beta} \left| B^e_1 \right|^2 \mathrm{Re}(\cos\theta_2), \tag{60}$$

(a) for incident P-wave

$$\langle P^{*e}_0 \rangle = -\frac{1}{2} \frac{\omega^4 \rho^e c_0^2}{\alpha} \left| A^e_0 \right| \cos\theta_0, \tag{61}$$

(b) for incident SV-wave

$$\langle P^{*e}_0 \rangle = -\frac{1}{2} \frac{\omega^4 \rho^e c_0^2}{\beta} \left| B^e_0 \right| \cos\theta_0 \tag{62}$$

are the average energy intensities of the reflected P-, reflected SV-, incident P-, and incident SV-waves respectively. In equation (59), negative sign is taken because the direction of reflected waves is opposite to that of incident wave.

For micropolar thermoelastic solid half-space with fractional order derivative, the average intensities of the waves on the surface with normal along x_3 - direction, are given by

$$\left\langle P_{ij}^{*}\right\rangle = \mathrm{Re}\left\langle t\right\rangle_{13}^{(i)}.\mathrm{Re}\left\langle \dot{u}_1^{(j)}\right\rangle + \mathrm{Re}\left\langle t\right\rangle_{33}^{(i)}.\mathrm{Re}\left\langle \dot{u}_3^{(j)}\right\rangle + \mathrm{Re}\left\langle m\right\rangle_{32}^{(i)}.\mathrm{Re}\left\langle \dot{\phi}_2\right\rangle^{(j)}. \tag{63}$$

The expressions for energy ratios for the transmitted LD-, transmitted T-, transmitted CDI- and transmitted CDII- waves are given by

$$E_{ij} = \frac{\left\langle P_{ij}^{*}\right\rangle}{\left\langle P_0^{*e}\right\rangle}, \qquad i, j = 1, 2, 3, 4, \tag{64}$$

where

$$\left\langle P_{ij}^{*}\right\rangle = -\frac{\omega^4}{2}\mathrm{Re}\left[\left\{(2\mu+K)\frac{dV_i}{\omega}\frac{\xi_R}{\omega}\frac{\overline{\xi}_R}{\omega} + \lambda\left(\frac{\xi_R}{\omega}\right)^2\left(\frac{d\overline{V}_j}{\omega}\right) + \rho c_1^2\left(\frac{dV_i}{\omega}\right)^2\left(\frac{d\overline{V}_j}{\omega}\right) + \right.\right.$$

$$\left.\left. + \frac{\rho n_i c_1^2}{\omega^2}\left(\frac{d\overline{V}_j}{\omega}\right)\left(1+\frac{(\tau_1)^{\alpha}}{\alpha!}(-i\omega)^{\alpha}\right)\right\}\frac{B_i\overline{B}_j}{\rho c_1^2}\right], \quad i, j = 1,2;$$

$$\left\langle P_{ij}^{*}\right\rangle = -\frac{\omega^4}{2}\mathrm{Re}\left[\left\{-(2\mu+K)\frac{dV_i}{\omega}\frac{d\overline{V}_j}{\omega}\frac{\xi_R}{\omega} + \lambda\left(\frac{\xi_R}{\omega}\right)^2\left(\frac{\overline{\xi}_R}{\omega}\right) + \rho c_1^2\left(\frac{dV_i}{\omega}\right)^2\left(\frac{\overline{\xi}_R}{\omega}\right) + \right.\right.$$

$$\left.\left. + \frac{\rho n_i c_1^2}{\omega^2}\left(\frac{\overline{\xi}_R}{\omega}\right)\left(1+\frac{(\tau_1)^{\alpha}}{\alpha!}(-i\omega)^{\alpha}\right)\right\}\frac{B_i\overline{B}_j}{\rho c_1^2}\right], \quad i = 1,2\ ;\ j = 3,4;$$

$$\left\langle P_{ij}^{*}\right\rangle = -\frac{\omega^4}{2}\mathrm{Re}\left[\left\{\left\{\mu\left(\left(\frac{\xi_R}{\omega}\right)^2 - \left(\frac{dV_i}{\omega}\right)^2\right) - K\left(\frac{dV_i}{\omega}\right)^2 + K\frac{n_i}{\omega^2}\right\}\frac{\overline{\xi}_R}{\omega} + (2\mu+K)\frac{\xi_R}{\omega}\frac{dV_i}{\omega}\frac{d\overline{V}_j}{\omega}\right\}\frac{\overline{B}_jB_i}{\rho c_1^2}\right],$$

$$i = 3,4;\ j = 1,2\ ;$$

$$\left\langle P_{ij}^{*}\right\rangle = -\frac{\omega^4}{2}\mathrm{Re}\left[\left\{\left\{\mu\left(\left(\frac{dV_i}{\omega}\right)^2 - \left(\frac{\xi_R}{\omega}\right)^2\right) + K\left(\frac{dV_i}{\omega}\right)^2 - K\frac{n_i}{\omega^2}\right\}\frac{d\overline{V}_j}{\omega} + \right.\right.$$

$$\left.\left. + (2\mu+K)\frac{\xi_R}{\omega}\frac{\overline{\xi}_R}{\omega}\frac{dV_i}{\omega} + \gamma\frac{\omega^{*2}}{c_1^2}\frac{n_i}{\omega}\frac{n_j}{\omega}\frac{dV_i}{\omega}\right\}\frac{B_i\overline{B}_j}{\rho c_1^2}\right], \quad i, j = 3,4\ .$$

The diagonal entries of energy matrix E_{ij} in equation (64) represents the energy ratios of LD, T, CDI, CDII waves respectively, whereas sum of the non diagonal entries of E_{ij} gives the share of interaction energy among all transmitted waves in the medium and is given by

$$E_{RR} = \sum_{i=1}^{4}\left(\sum_{j=1}^{4}E_{ij} - E_{ii}\right). \tag{65}$$

The energy ratios E_i, i =1, 2, diagonal entries and non diagonal entries of energy matrix E_{ij}, that is, E_{11}, E_{22}, E_{33}, E_{44} and E_{RR} yield the conservation of incident energy across the interface, through the relation

$$E_1 + E_2 + E_{11} + E_{22} + E_{33} + E_{44} + E_{RR} = 1. \tag{66}$$

6. Numerical results and discussion

We now represent some numerical results, the physical data for which is given below:

$$\lambda = 9.4\times10^{11}\ \text{Kg m}^{-1}\text{s}^{-2},\quad \mu = 4\times10^{11}\ \text{Kg m}^{-1}\text{s}^{-2},\quad T_0 = 298\ \text{K},\quad j = 0.2\times10^{-15}\ \text{m}^2,$$
$$\rho = 8.954\times10^{3}\ \text{Kg m}^{-3},\quad C^* = 0.3831\times10^{3}\ \text{J Kg}^{-1}\text{K}^{-1},\quad \alpha_t = 1.78\times10^{-5}\ \text{K}^{-1},\quad h = 0,$$
$$K = 1.0\times10^{11}\ \text{N/m}^2,\quad K^* = 0.0017\times10^{-2}\ \text{W m}^{-1}\text{K}^{-1},\quad \gamma = 0.779\times10^{-4}\ \text{N},$$
$$a_5 = 0.073,\quad \tau_0 = 6.131\times10^{-13}\ \text{s},\quad \tau_1 = 8.765\times10^{-13}\ \text{s}.$$

Following Bullen [40], the numerical data of granite in elastic medium is given by

$$\rho^e = 20\times10^{3}\ \text{Kg m}^{-3},\quad \alpha^e = 5.27\times10^{3}\ \text{m s}^{-1},\quad \beta^e = 3.17\times10^{3}\ \text{m s}^{-1}.$$

The software Matlab 7.0.4 has been used to determine the values of energy ratios E_i, $i = 1,2$ and an energy matrix E_{ij}, $i, j = 1,2,3,4$, defined in previous Section for different values of incident angle (θ_0) ranging from 0 to 90° for fixed frequency $\omega = 2\times\pi\times0.5$ Hz. corresponding to incident P and SV waves, the variation of these energy ratios with respect to angle of incidence have been plotted in Figs. 2-8 and Figs. 9-15 respectively. In all the figures, the slant and horizontal lines correspond to Lord Shulman theory (LS) for fractional order α=1 and α=2 respectively whereas horizontal and slant boxes correspond to Green Lindsay theory (GL) for fractional order α=1 and α=2 respectively. To show the effect of different theories of thermoelasticity i.e. for LS, GL theories graphs are drawn in 3D.

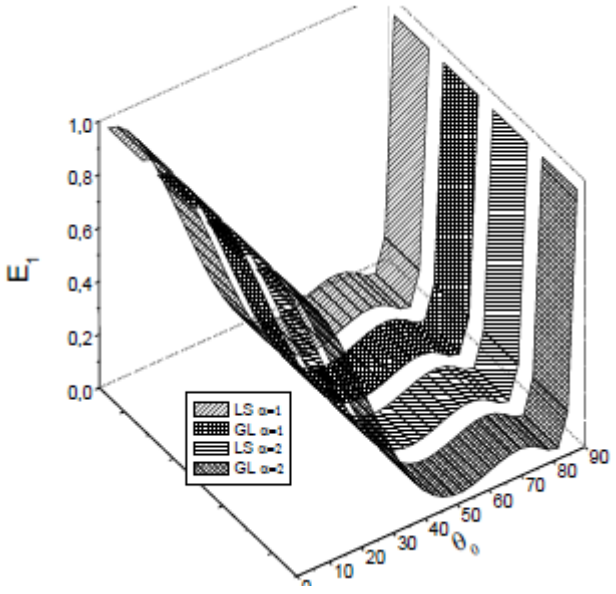

Fig. 2. Variation of energy ratio E_1 w.r.t. angle of incidence θ_0 for P wave.

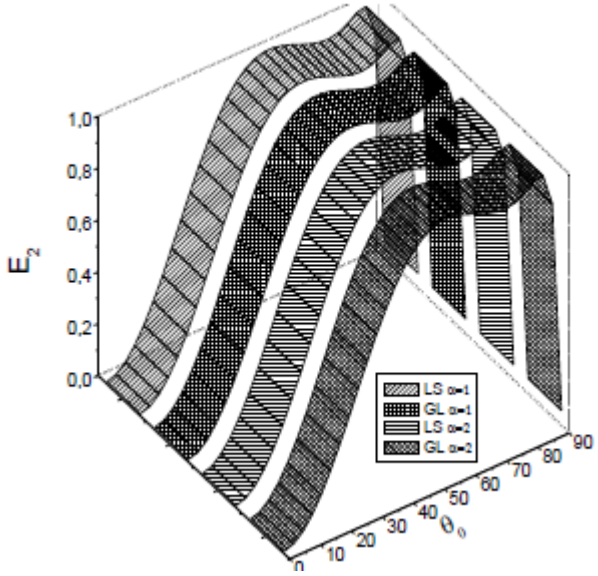

Fig. 3. Variation of energy ratio E_2 w.r.t. angle of incidence θ_0 for P wave.

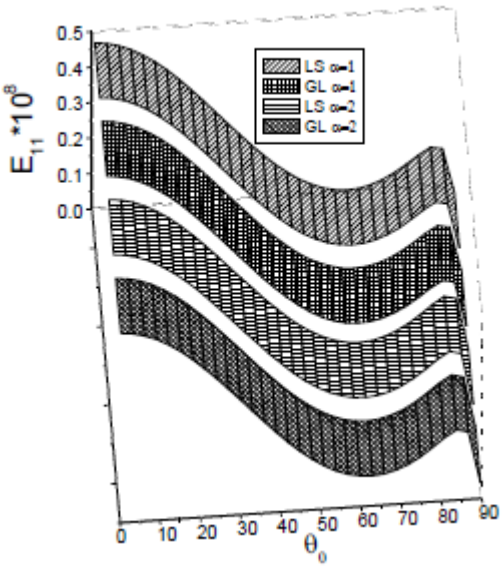

Fig. 4. Variation of energy ratio E_{11} w.r.t. angle of incidence θ_0 for P wave.

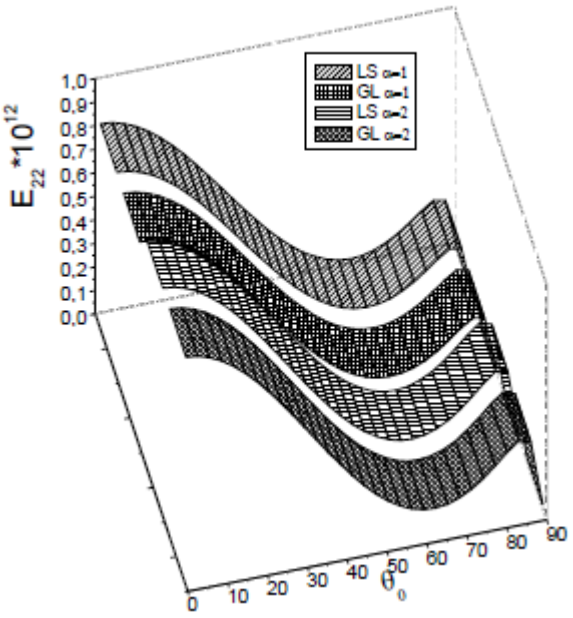

Fig. 5. Variation of energy ratio E_{22} w.r.t. angle of incidence θ_0 for P wave.

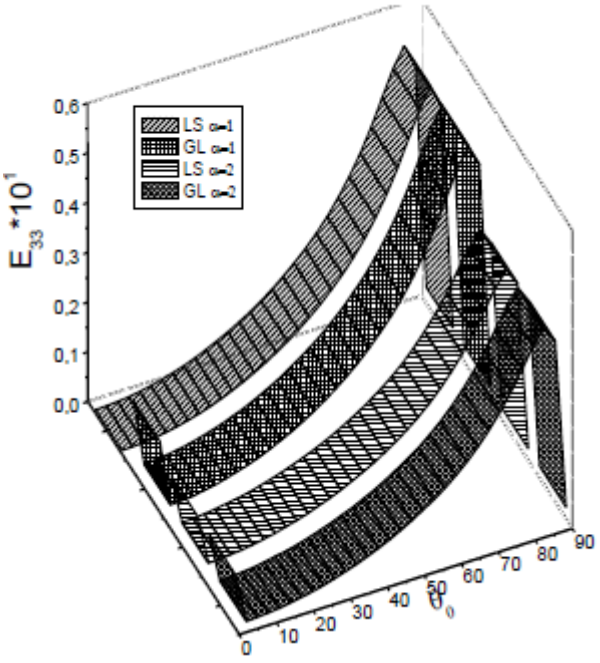

Fig. 6. Variation of energy ratio E_{33} w.r.t. angle of incidence θ_0 for P wave.

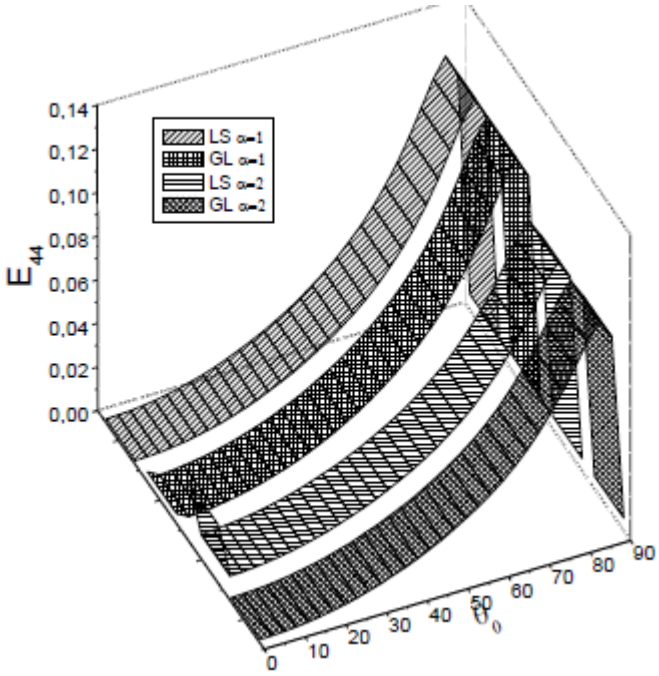

Fig. 7. Variation of energy ratio E_{44} w.r.t. angle of incidence θ_0 for P wave.

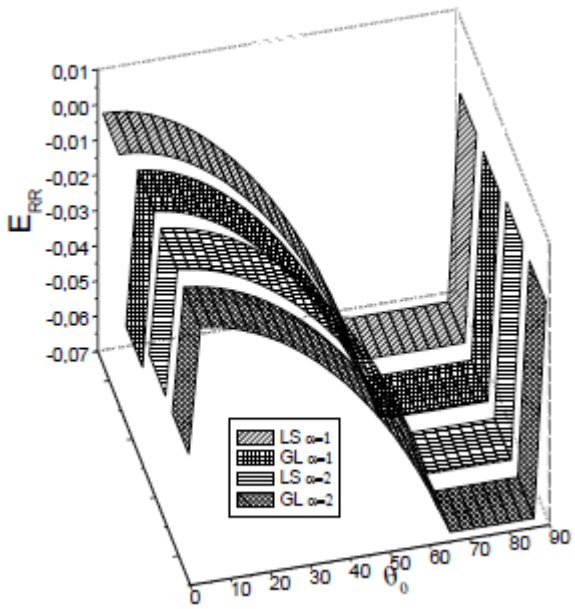

Fig. 8. Variation of energy ratio E_{RR} w.r.t. angle of incidence θ_0 for P wave.

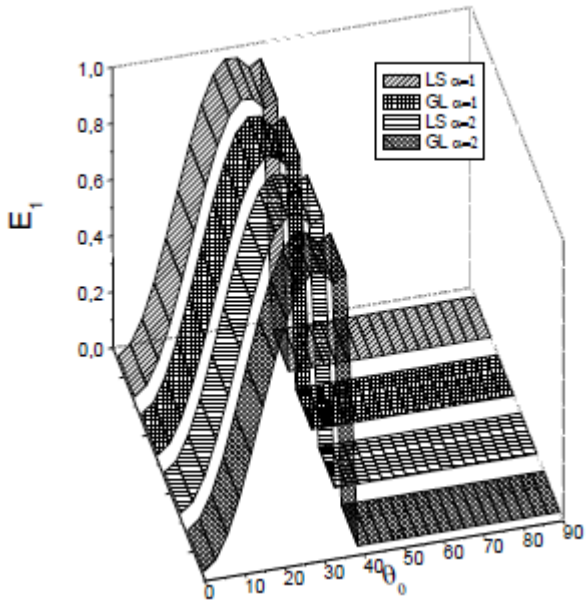

Fig. 9. Variation of energy ratio E_1 w.r.t. angle of incidence θ_0 for SV wave.

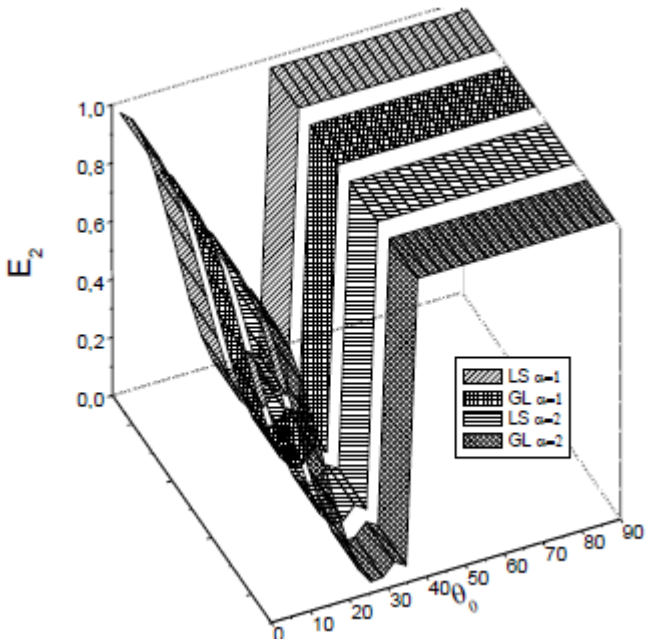

Fig. 10. Variation of energy ratio E_2 w.r.t. angle of incidence θ_0 for SV wave.

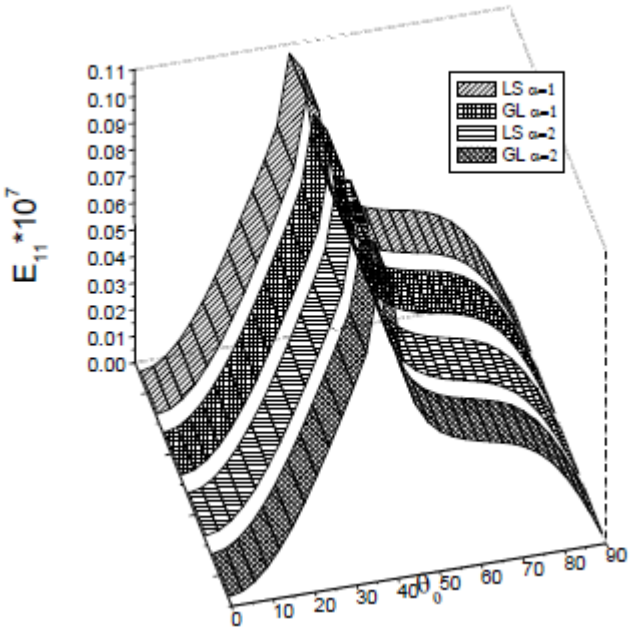

Fig. 11. Variation of energy ratio E_{11} w.r.t. angle of incidence θ_0 for SV wave.

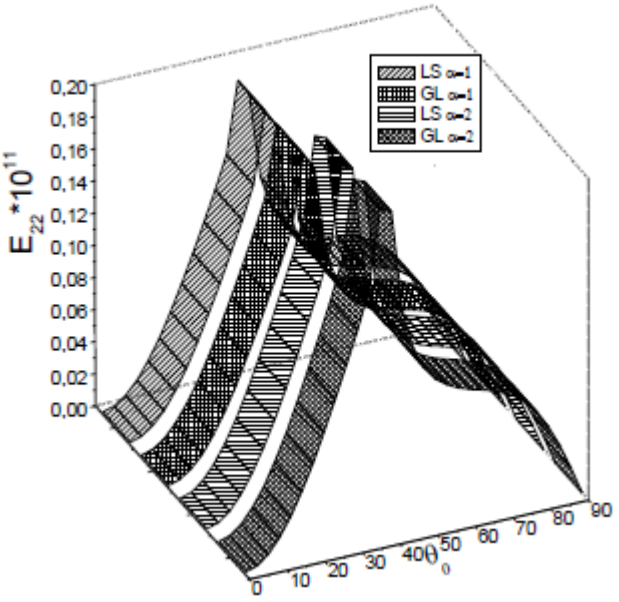

Fig. 12. Variation of energy ratio E_{22} w.r.t. angle of incidence θ_0 for SV wave.

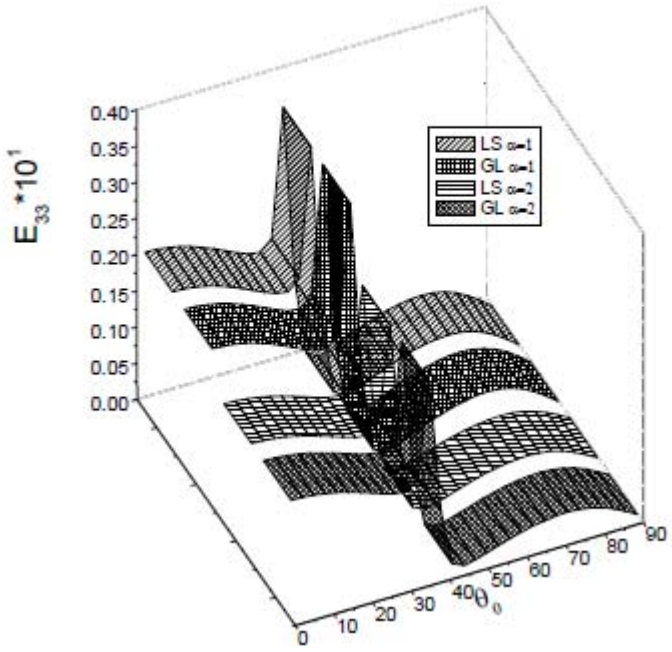

Fig. 13. Variation of energy ratio E_{33} w.r.t. angle of incidence θ_0 for SV wave.

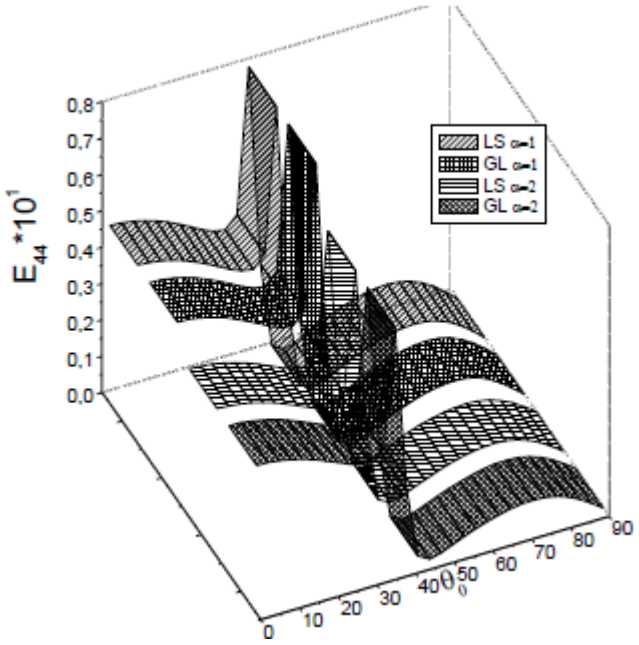

Fig. 14. Variation of energy ratio E_{44} w.r.t. angle of incidence θ_0 for SV wave.

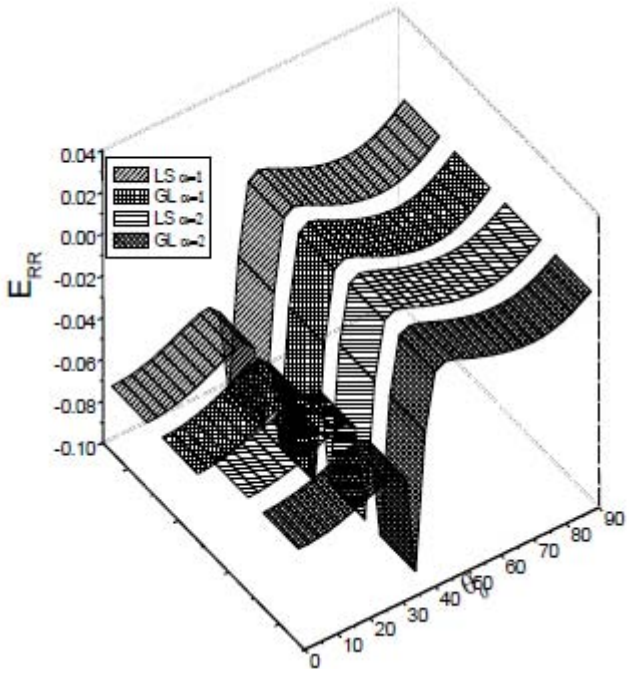

Fig. 15. Variation of energy ratio E_{RR} w.r.t. angle of incidence θ_0 for SV wave.

Incident P wave. It is clear from Fig. 2 that for both LS and GL theories, the values of energy ratio E_1 first decrease smoothly with increase of angle of incidence (θ_0) from 0 to 50° and then fluctuate in the range $50^\circ \le \theta_0 \le 80^\circ$ $\theta_0$and increase as θ_0 increases further for both values of α. From Fig. 3 the behavior and variation of E_2 is opposite that of E_1 for LS and GL theories. Figure 4 indicates that for the value of energy ratio E_{11} decrease in the range $0 \le \theta_0 \le 60^\circ$ and increase thereafter for both LS and GL theories values of fractional order derivative α. But the magnitude values in case of GL theory are higher in comparison to LS theory. Figure 5 depicts that E_{22} shows the same behavior and variation as E_{11} but with different magnitude values. But the magnitude values of E_{22} in case of fractional derivative α=2 are higher. Figure 6 depicts that the values of E_{33} increase smoothly for $0^\circ \le \theta_0 \le 80^\circ$ and decrease as θ_0 increases further. The values of energy ratio E_{33} are higher for fractional order α=1 except the range $0 \le \theta_0 \le 5^\circ$ where E_{33} attains minimum value for LS (α=1). Figure 7 depicts that the trend and variation of E_{44} is similar as E_{33} for $\theta_0 \ge 5^\circ$ but with different magnitude values. Figure 8 shows that E_{RR} increases in the range $0 \le \theta_0 \le 10^\circ$ and decreases for $10^\circ \le \theta_0 \le 70^\circ$ and attain small value in the range $70^\circ \le \theta_0 \le 80^\circ$ and further increase rapidly. Values of energy ratio E_{RR} are higher for fractional order α=2. It is noticed that the sum of the values of energy ratios E_1, E_2, E_{11}, E_{22}, E_{33}, E_{44} and E_{RR} is found to be exactly unity at each value of θ_0 which proves the law of conservation of energy at the interface.

Incident SV wave. From Fig. 9 it is noticed that for LS and GL theories and for both fractional orders E_1 increase for $0 \le \theta_0 \le 25^\circ$, fluctuate for $25^\circ \le \theta_0 \le 35^\circ$ and decrease up to $\theta_0 = 40^\circ$ and attain minimum value in the range $40^\circ \le \theta_0 \le 90^\circ$. Figure 10 depicts the behavior of E_2 opposite to that of E_1 for LS, GL theories with difference in the magnitude values for both fractional orders and has maximum value 1 in the range $40^\circ \le \theta_0 \le 90^\circ$. From Fig. 11 and Fig. 12 it is noticed that E_{11} and E_{22} depict the same behavior and variation but magnitude values of E_{11} are higher than E_{22}. Both attain maximum value in the range $30^\circ \le \theta_0 \le 40^\circ$. But the value of energy ratio E_{11} are higher for fractional order derivative α=1 whereas E_{22} attains higher value for α=2. Figure 13 indicates that E_{33} attains maximum value at $\theta_0 = 36^\circ$. Figure 14 indicates that E_{44} shows the same behavior and variation as E_{33} but with different magnitude values. The magnitude value of E_{33} and E_{44} are higher for fractional order α=1. The values of energy ratio E_{44} are higher in comparison to E_{33}. From Fig. 15 it is evident that the values of energy ratio E_{RR} are minimums for LS, GL theories and for both fractional orders in the range $30^\circ \le \theta_0 \le 50^\circ$, then increases and finally attains the steady value. The value of E_{RR} are higher for fractional order α=2 for both the theories. Like in case of incident P wave, the sum of all energy ratios is also found to be unity in the case of incident SV wave. The sum of all the interaction energies is exactly one but it does not looks to be 1 from 3D graphs, but if we draw in any other software it looks to be 1.

8. Conclusions

In the present article, the phenomenon of reflection and transmission of obliquely incident elastic wave at the interface between an elastic solid half-space and a micropolar thermoelastic solid half-space with fractional order derivative has been studied. The four waves in micropolar thermoelastic solid medium are identified and explained through different wave equations in terms of displacement potentials. The energy ratios of different reflected and refracted waves to that of incident wave are computed numerically and presented graphically with respect to the angle of incidence for LS and GL theories and fractional orders α=1 and α=2. From numerical results, we conclude that the effect of fractional order derivatives and diffusion on the energy ratios of the reflected and refracted waves is significant. The value of energy ratios E_{11}, E_{33}, E_{44} are higher for fractional order derivative α=1 whereas values of energy ratio E_{22} are higher for fractional order derivative α=2. The sum of all energy ratios of the reflected waves, refracted waves and interference between refracted waves is verified to be always unity which ensures the law of conservation of incident energy at the interface.

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