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IDENTIFICATION OF DEFECTS IN A SOLID BODY ON THE BASE OF SURFACE DISPLACEMENTS

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Abstract. The aim of our work is to discuss how the surface displacements caused by an applied force can be used for the identification of defects placed in near-surface layers of the body. As an illustrative example of the possibility for such identification, the elastic problem for the half-space weakened by a circular hole is considered. First of all we present the complete and correct analytical solution of the plane elasticity problem for the concentrated force acting on the surface of a half-space with a hole. We describe the biharmonic stress-function used for the derivation of stresses and strains in the half-space with a hole and the associated biharmonic function that allows to determine the displacement field. Both functions are given in the form of Fourier series with the compact coefficients. It is shown that the found analytical formulas of surface displacements give the way to find the circular hole diameter and position when the applied force and elastic modules of the material are known.

1. Introduction

The displacements at the surface of an elastic body induced by an applied force depend on the state of near-surface material of the body and can be sensitive to defects of various natures that are present in the surface vicinity. It is expected that with the help of surface displacements caused by applied force, one can determine the parameters of the subsurface defects. These defects are *inhomogeneities, voids* and *cracks* (Fig. 1). Thus, the information on how the defect affects the surface displacement induced by the given force is required for elaboration of approaches to the defect identification. This is the subject of the inverse problem for the determining of defect parameters from known surface displacement fields generated by the given force. The modern tools used by the indentation technique [1] make it possible the precise measurement of the surface displacements. Practical motivation for investigation of surface displacements is related to the progress in use of tactile sensors (see review [2]). Such sensors, e.g. biologically inspired sensors, are able to detect force in a specific direction, in particular when the force is acting on the surface of an elastic body. Additional equipment allows simultaneous measurements of the displacements at various points at the surface of this body.

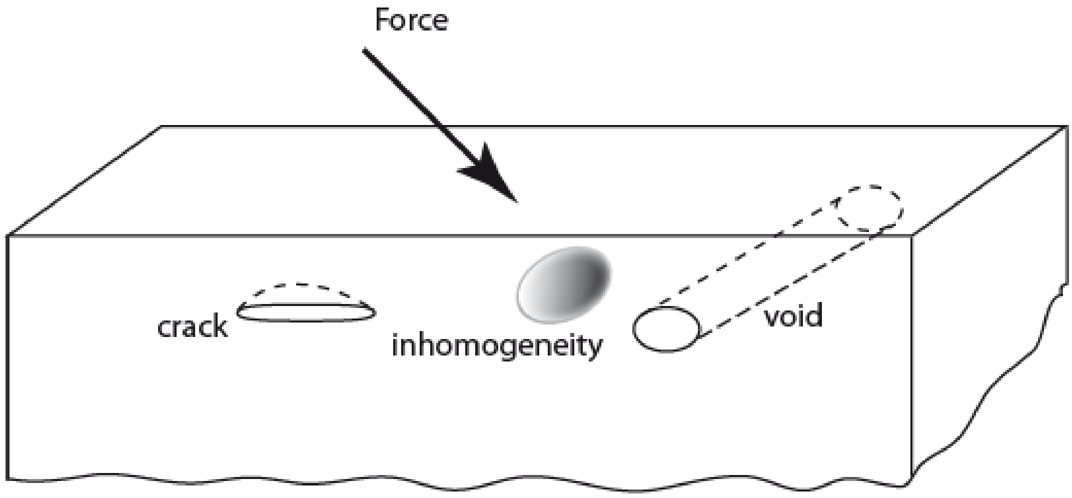

Fig. 1. Near-surface layer of the body with *inhomogeneity, void* and *crack.*

One cannot state the inverse problem without having in hands the solution of the direct problem. In the following we focus on the direct problem in the simplest case of a plane elasticity problem for an isotropic half-space with a cylindrical hole. To approach this problem the use of bipolar coordinate system is convenient [3, 4].

The algorithm for solving the plane elasticity problems with the geometry corresponding to the bipolar coordinate system, in particular, the modified algorithm for the half-space with a hole, was given G.B. Jeffery as early as in 1920 [3]. In 1962, R.M. Evan-Iwanowski published the stress functions for this problem [5]. The author applied the algorithm by G.B. Jeffery and considered the normal and tangential forces acting on the boundary of the half-space. However in practice, the use of the solution from Ref. [5] is not possible, because of the evident mistakes in published formulas. Later, this problem has been solved in other ways applying numerical methods and has been extended, for example, to the case with elliptical hole (see, *e.g.* [6]). In any case, published and known solutions do not allow anyone to determine the displacement field in the loaded half-space with a hole, in particular, the displacements at the surface of the half-space.

In order to get valid analytical stress functions and elastic fields and, in particular, to determine the surface displacements, we re-investigated this classical elasticity problem.

2. Plane elasticity solution for a half-space weakened by a circular hole and loaded by a concentrated force

2.1. Statement of the problem, geometry and basic equations. Let us consider an elastic half-space $y \geq 0$ with an inner circular cylindrical hole. The linear force $\mathbf{f} = f_T\mathbf{e}_x + f_N\mathbf{e}_y$ is applied at the point $(x_0, 0)$ (Fig. 2). It is required to find the analytical solution of this elasticity boundary value problem, i.e. to determine biharmonic stress function and the elastic fields in the half-space with hole and to determine the displacements at the half-space boundary.

A convenient coordinate system to get solutions for elasticity problem in such a geometry is bipolar coordinate system (α, β). Relationship among bipolar (α, β) coordinates, Cartesian coordinates (x, y), and polar radius r are listed below.

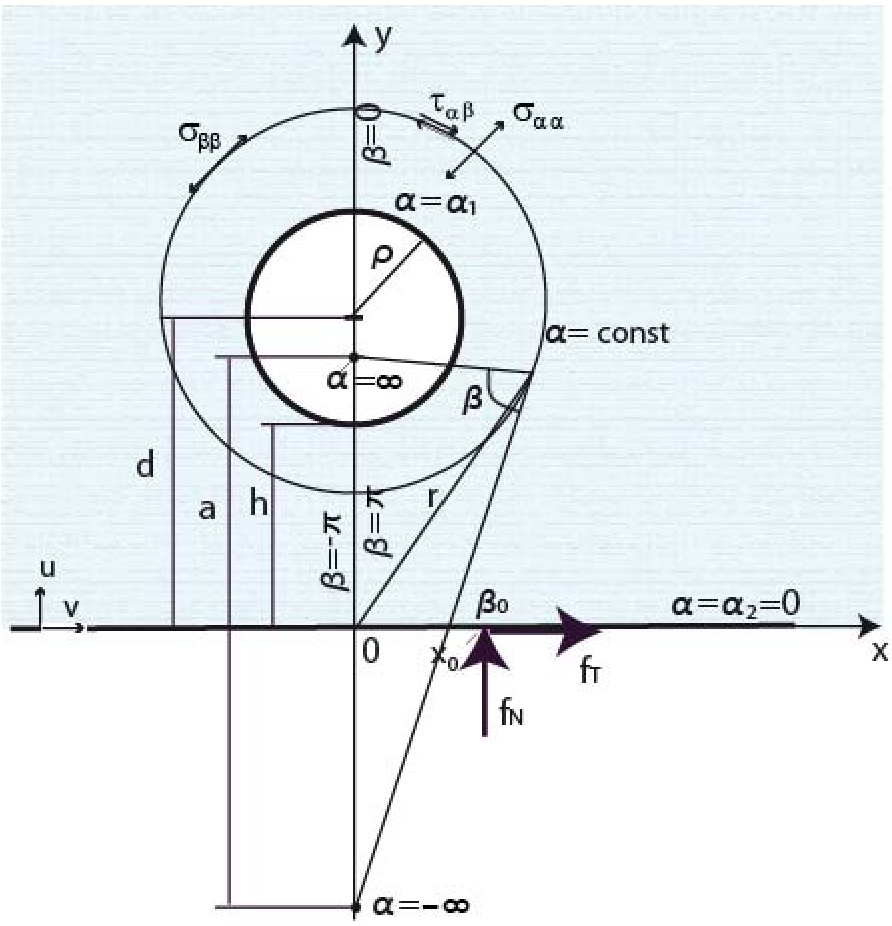

Fig. 2. An elastic half-space $y \geq 0$ containing a circular hole and loaded by linear force $\mathbf{f} = f_T \mathbf{e}_x + f_N \mathbf{e}_y$. The bipolar (α, β) and Cartesian (x, y) coordinate systems are shown. The hole and the boundary of a half-space have bipolar coordinates $\alpha = \alpha_1$ and $\alpha = \alpha_2 = 0$, correspondingly. Force is applied at the point $(0, \beta_0)$ that corresponds to the Cartesian coordinates $(x_0, 0)$.

We also give useful formulas relating radius of the hole ρ, the coordinate of the center of the hole d, their aspect ratio $\delta = d / \rho$, and the shortest distance between boundary of the hole and half-space boundary h with the bipolar coordinate of the hole α_1 $(\alpha_1 > 0)$:

$$\frac{x}{a}=\frac{\sin\beta}{\cosh\alpha-\cos\beta}, \qquad \frac{y}{a}=\frac{\sinh\alpha}{\cosh\alpha-\cos\beta}, \tag{1a,b}$$

$$\frac{r^2}{a^2}=\frac{\cosh\alpha+\cos\beta}{\cosh\alpha-\cos\beta}, \tag{1c}$$

$$\rho=\frac{a}{\sinh\alpha_1}, \quad d=a\coth\alpha_1, \quad \delta=\frac{d}{\rho}=\cosh\alpha_1, \quad h=a\tanh\tfrac{\alpha_1}{2}. \tag{1d,e,f,g}$$

Here $-\infty<\alpha<\infty$, $-\pi\le\beta\le\pi$. The discontinuity of coordinate β takes place on the segment $(-a,+a)$ of axis $0y$ (Fig. 2 shows part of the segment $(-a,+a)$ along the positive direction of the axis y-axis). Above the point $+a$ and below the point $-a$ along the y-axis bipolar coordinate $\beta=0$. The axis $0x$ is the line $\alpha=0$, the Cartesian points $(0,\pm a)$ correspond to $\alpha=\pm\infty$. The point at infinity has coordinate $(\alpha=0, \beta=0)$.

In the bipolar coordinate system stresses ($\sigma_{\alpha\alpha}, \sigma_{\beta\beta}, \tau_{\alpha\beta}$) and strains ($\varepsilon_{\alpha\alpha}, \varepsilon_{\beta\beta}, \varepsilon_{\alpha\beta}$) can be found from the biharmonic function Φ; and the displacements (u, v) are obtained by using two interconnected biharmonic functions - main function Φ and associated function Ψ [3].

In this representation the stresses are given as following [3]:

$$a\sigma_{\alpha\alpha}=\left[(\cosh\alpha-\cos\beta)\frac{\partial^2}{\partial\beta^2}-\sinh\alpha\frac{\partial}{\partial\alpha}-\sin\beta\frac{\partial}{\partial\beta}+\cosh\alpha\right](g\Phi), \tag{2a}$$

$$a\sigma_{\beta\beta}=\left[(\cosh\alpha-\cos\beta)\frac{\partial^2}{\partial\alpha^2}-\sinh\alpha\frac{\partial}{\partial\alpha}-\sin\beta\frac{\partial}{\partial\beta}+\cos\beta\right](g\Phi), \tag{2b}$$

$$a\tau_{\alpha\beta}=-(\cosh\alpha-\cos\beta)\frac{\partial^2}{\partial\alpha\partial\beta}(g\Phi), \tag{2c}$$

where $g=\dfrac{\cosh\alpha-\cos\beta}{a}$.

The strains can be determined through the Hooke's law.

The relations between functions Φ and Ψ are defined as [3]:

$$\frac{\partial^2}{\partial\alpha\,\partial\beta}(g\Psi)=\frac{\lambda+2\mu}{2(\lambda+\mu)}\left[\frac{\partial^2}{\partial\alpha^2}-\frac{\partial^2}{\partial\beta^2}-1\right](g\Phi), \tag{3a}$$

$$g\Psi=\frac{\lambda+2\mu}{2(\lambda+\mu)}\iint\left[\frac{\partial^2}{\partial\alpha^2}-\frac{\partial^2}{\partial\beta^2}-1\right](g\Phi)\,d\alpha d\beta, \tag{3b}$$

where μ, λ are Lame constants.

The displacements have the following forms in terms of the biharmonic functions Φ and Ψ [3]:

$$u = \frac{g}{2\mu}\left[\frac{\mu}{\lambda+\mu}\cdot\frac{\partial\Phi}{\partial\alpha} - \frac{\partial\Psi}{\partial\beta}\right], \tag{4a}$$

$$v = \frac{g}{2\mu}\left[\frac{\mu}{\lambda+\mu}\cdot\frac{\partial\Phi}{\partial\beta} + \frac{\partial\Psi}{\partial\alpha}\right], \tag{4b}$$

where u, v are displacements in the directions normal to the lines of constant coordinates α and β, respectively [3], in other words $u \equiv u_\alpha$, $v \equiv u_\beta$.

According to Eq. (3b) associated biharmonic function Ψ is determined up to the accuracy of some functions of α and β. As correctly pointed J.B. Jeffery [3], the only possible arbitrary terms in $g\Psi$, that do not affect the stresses, are given by:

$$g\Psi = Aa(\cosh\alpha + \cos\beta) + B(\cosh\alpha - \cos\beta) + Ca\sinh\alpha + Da\sin\beta \tag{5a}$$

or in Cartesian coordinates

$$\Psi = Ar^2 + aB + Cy + Dx\,. \tag{5b}$$

The terms in Eq. (5a,b) (except the second one) correspond to rotation about the origin and pure translation of a rigid body [3].

Strains can be determined from the displacements Eq. (4) [3]:

$$\varepsilon_{\alpha\alpha} = g\frac{\partial u}{\partial\alpha} - v\frac{\partial g}{\partial\beta}, \tag{6a}$$

$$\varepsilon_{\beta\beta} = g\frac{\partial v}{\partial\beta} - u\frac{\partial g}{\partial\alpha}; \tag{6b}$$

$$\varepsilon_{\alpha\beta} = \frac{\partial}{\partial\alpha}(gv) + \frac{\partial}{\partial\beta}(gu)\,. \tag{6c}$$

Hooke's law allows one to define stresses:

$$\sigma_{\alpha\alpha} = (\lambda+2\mu)\,\varepsilon_{\alpha\alpha} + \lambda\,\varepsilon_{\beta\beta}, \quad \sigma_{\beta\beta} = (\lambda+2\mu)\,\varepsilon_{\beta\beta} + \lambda\,\varepsilon_{\alpha\alpha}, \quad \tau_{\alpha\beta} = \mu\varepsilon_{\alpha\beta}\,. \tag{7a,b,c}$$

A comparison of stresses obtained with the formulas Eq. (2) and Eq. (7) helps to verify the displacements, calculated from Eq. (4).

2.2. Biharmonic function Φ and stresses. The required biharmonic function Φ is sought as a sum of the known function $\Phi^0_{N,T}$, giving the solution for normal ($f_N\mathbf{e}_y$) or tangential ($f_T\mathbf{e}_x$) force applied at the uniform (i.e. hole-less) half-space boundary, and an additional function $\Phi^*_{N,T}$ due to the presence of the hole:

$$\Phi_{N,T} = \Phi^0_{N,T} + \Phi^*_{N,T}\,. \tag{8}$$

Biharmonic functions $\Phi^0_{N,T}$ have the following standard forms [7]:

$$\Phi^0_N = -\frac{f_N}{\pi}(x-x_0)\arctan\left(\frac{x-x_0}{y}\right), \tag{9a}$$

$$\Phi^0_T = -\frac{f_T}{\pi} y \ \arctan\left(\frac{x-x_0}{y}\right), \tag{9b}$$

where x_0 is a coordinate of the applied force.

The stresses corresponding to the function $\Phi^0_{N,T}$ satisfy conditions at the planar boundary of the half-space:

$$\sigma^0_{\alpha\alpha}\Big|_{\alpha=0} = 0\,, \quad \tau^0_{\alpha\beta}\Big|_{\alpha=0} = 0 \tag{10a,b}$$

It is therefore required to find $\Phi^*_{N,T}$, which proves the stresses satisfying the boundary conditions:

$$\sigma^*_{\alpha\alpha}\Big|_{\alpha=0} = 0, \qquad \tau^*_{\alpha\beta}\Big|_{\alpha=0} = 0, \tag{11a,b}$$

$$\sigma^0_{\alpha\alpha}\Big|_{\alpha=\alpha_1} + \sigma^*_{\alpha\alpha}\Big|_{\alpha=\alpha_1} = 0, \qquad \tau^0_{\alpha\beta}\Big|_{\alpha=\alpha_1} + \tau^*_{\alpha\beta}\Big|_{\alpha=\alpha_1} = 0, \tag{11c,d}$$

where α_1 is a coordinate of the free surface of the hole (Fig. 2).

The general expressions for the biharmonic functions $\Phi^*_{N,T}$, which give the stresses that satisfy the conditions at the straight boundary Eq. (11a,b) are known [3, 4]. In order not to overload the subscripts N and T the formulas below we write the expression for the biharmonic functions $\Phi^*_{N,T}$ as following:

$$g\Phi^* = (\cosh\alpha - \cos\beta)B_0\alpha + \sum_{n=1}^{\infty}[\phi^c_n(\alpha)\cos n\beta + \phi^s_n(\alpha)\sin n\beta], \tag{12}$$

$$\phi^c_1 = A^c_1\cosh 2\alpha - A^c_1 + \tfrac{1}{2}B_o\sinh 2\alpha\,,$$

$$\phi^c_n = A^c_n\cosh(n+1)\alpha - A^c_n\cosh(n-1)\alpha + C^c_n\sinh(n+1)\alpha - \frac{(n+1)}{(n-1)}C^c_n\sinh(n-1)\alpha, \quad (n\ge 2)\,,$$

$$\phi^s_1 = A^s_1\cosh 2\alpha\,,$$

$$\phi^s_n = A^s_n\cosh(n+1)\alpha - A^s_n\cosh(n-1)\alpha + C^s_n\sinh(n+1)\alpha - \frac{(n+1)}{(n-1)}C^s_n\sinh(n-1)\alpha, \quad (n\ge 2)\,.$$

It can be easily demonstrated that the stress function Φ^* Eq. (12) with arbitrary coefficients $B_0, A_n^c, C_n^c, A_n^s, C_n^s$ gives the stresses Eq. (2) satisfying the boundary conditions Eq. (11a,b) automatically [3].

The algorithm to find the coefficients $B_0, A_n^c, C_n^c, A_n^s, C_n^s$ are defined as follows: in the boundary condition equations Eqs. (11c, d) the known stresses $\sigma_{\alpha\alpha}^0\big|_{\alpha=\alpha_1}$ and $\tau_{\alpha\beta}^0\big|_{\alpha=\alpha_1}$ are represented in the form of Fourier series with respect to the variable β, and the unknown $\sigma_{\alpha\alpha}^*\big|_{\alpha=\alpha_1}$ and $\sigma_{\alpha\alpha}^*\big|_{\alpha=\alpha_1}$ are expressed with the help of relations Eq. (2) through the biharmonic function Φ^* in the form of Eq. (12). Equations (11c,d) are solved for the unknown coefficients of the series of Eq. (12) (about general algorithm see Ref. [3, 4]).

Fourier series of stresses $\sigma_{\alpha\alpha}^0\big|_{\alpha=\alpha_1}$ and $\tau_{\alpha\beta}^0\big|_{\alpha=\alpha_1}$ and the detailed algorithm for determining the coefficients $B_0, A_n^c, C_n^c, A_n^s, C_n^s$ are given for the normal and tangential forces applied at the planar boundary in our article [7].

Then stresses σ_{ij} under consideration are the sum of stresses σ_{ij}^0 caused by the force in the uniform half-space and the additional stresses σ_{ij}^* due to the hole:

$$\sigma_{\alpha\alpha} = \sigma_{\alpha\alpha}^0 + \sigma_{\alpha\alpha}^*, \quad \sigma_{\beta\beta} = \sigma_{\beta\beta}^0 + \sigma_{\beta\beta}^*, \quad \tau_{\alpha\beta} = \tau_{\alpha\beta}^0 + \tau_{\alpha\beta}^*. \qquad \text{(13a,b,c)}$$

The terms in the sums Eqs. (13a,b,c) are calculated from the known stress functions $\Phi_{N,T}^0$ Eqs. (9a,b) and the additional stress functions $\Phi_{N,T}^*$ Eq. (12) on the basis of relations Eqs. (2a,b,c).

2.3. Biharmonic function Ψ and displacements. The displacements u, v for the considered geometry are determined as a sum of the displacements in an uniform half-space u^0, v^0 and the additional displacements caused by the hole u^*, v^*:

$$u = u^0 + u^*, \qquad \text{(14a)}$$

$$v = v^0 + v^*, \qquad \text{(14b)}$$

where, as before, u, v are the displacements in the directions normal to the lines of constant α and β, respectively [3],

In order to derive the displacements u^*, v^* from Eq. (4) in addition to the function Φ^* it is necessary to know the associated biharmonic function Ψ^*. This function is constructed on the base of the biharmonic function Φ^* [3]:

$$g\Psi^* = -\left(\frac{\lambda+2\mu}{\lambda+\mu}\right)\left\{(-\cosh\alpha+\cos\beta)B_0\beta + \sum_{n=1}^{\infty}[\psi_n^c(\alpha)\cos n\beta + \psi_n^s(\alpha)\sin n\beta]\right\} - g\Psi^{*add}, \quad (15)$$

where

$$\psi_1^c = A_1^s \sinh 2\alpha,$$

$$\psi_n^c = A_n^s \sinh(n+1)\alpha - A_n^s \sinh(n-1)\alpha + C_n^s \cosh(n+1)\alpha - \tfrac{(n+1)}{(n-1)} C_n^s \cosh(n-1)\alpha, \quad (n \ge 2)$$

$$\psi_1^s = -A_1^c \sinh 2\alpha - \tfrac{1}{2} B_o \cosh 2\alpha ,$$

$$\psi_n^s = -(A_n^c \sinh(n+1)\alpha - A_n^c \sinh(n-1)\alpha + C_n^c \cosh(n+1)\alpha - \tfrac{(n+1)}{(n-1)} C_n^c \cosh(n-1)\alpha), \quad (n \ge 2)$$

where the coefficients $B_0, A_n^c, C_n^c, A_n^s, C_n^s$ are the same as in the expression for the function Φ^* of Eq. (12).

We added special term $-g\Psi^{*add}$ in Eq. (15) to eliminate the rotation and pure translation of the body as a whole (see Eq. (5)):

$$g\Psi^{*add} = a(\cosh\alpha + \cos\beta)\sum_{n=1}^{\infty} A_n^{add} + a\sinh\alpha \sum_{n=1}^{\infty} C_n^{add} + a\sin\beta \sum_{n=1}^{\infty} D_n^{add} \ . \tag{16}$$

Here the coefficients $A_n^{add}, C_n^{add}, D_n^{add}$ are found from the condition of vanishing displacements at infinity $\alpha = 0, \beta = 0$ [8].

2.4. Concentrated force with a magnitude of f_N is normal to the planar boundary. For the normal force acting on the straight boundary of the half-space, the stress function Φ_N^* caused by the hole is represented by the Fourier series Eq. (12) with the following coefficients:

$$B_0 = f_N \frac{(\cos\beta_0 - 1)\sinh 2\alpha_1}{\pi(\cosh 2\alpha_1 - 1)^2}, \tag{17a}$$

$$A_1^c = f_N \frac{1 - \cos\beta_0}{2\pi(\cosh 2\alpha_1 - 1)}, \tag{17b}$$

$$A_n^c = -f_N \frac{\sinh^2\alpha_1}{\pi(\sinh^2 n\alpha_1 - n^2\sinh^2\alpha_1)} \cos\tfrac{n\beta_0}{2}(n\cos\tfrac{n\beta_0}{2} - \cot\tfrac{\beta_0}{2}\sin\tfrac{n\beta_0}{2}), \quad (n \ge 2) \tag{17c}$$

$$C_n^c = f_N \frac{[n\sinh\alpha_1(\cosh\alpha_1 + n\sinh\alpha_1) + e^{-n\alpha_1}\sinh n\alpha_1]}{\pi n(1+n)(\sinh^2 n\alpha_1 - n^2\sinh^2\alpha_1)} \cos\tfrac{n\beta_0}{2}(n\cos\tfrac{n\beta_0}{2} - \cot\tfrac{\beta_0}{2}\sin\tfrac{n\beta_0}{2}), \quad (n \ge 2), \tag{17d}$$

$$A_1^s = 0, \tag{17e}$$

$$A_n^s = A_n^c \tan\tfrac{n\beta_0}{2} = -f_N \frac{\sinh^2\alpha_1}{\pi(\sinh^2 n\alpha_1 - n^2\sinh^2\alpha_1)} \sin\tfrac{n\beta_0}{2}(n\cos\tfrac{n\beta_0}{2} - \cot\tfrac{\beta_0}{2}\sin\tfrac{n\beta_0}{2}), \quad (n \ge 2), \tag{17f}$$

$$C_n^s = C_n^c \tan\tfrac{n\beta_0}{2} =$$

$$= f_N \frac{[n \sinh \alpha_1 (\cosh \alpha_1 + n \sinh \alpha_1) + e^{-n\alpha_1} \sinh n\alpha_1]}{\pi n (1+n)(\sinh^2 n\alpha_1 - n^2 \sinh^2 \alpha_1)} \sin \tfrac{n\beta_0}{2} (n \cos \tfrac{n\beta_0}{2} - \cot \tfrac{\beta_0}{2} \sin \tfrac{n\beta_0}{2}),$$

$$(n \geq 2) \qquad (17g)$$

The stress function Φ_N^*, given by means of expressions Eq.(12) and Eq.(17), has quite a compact form and allows to find the stresses σ_{ij}^* and strains ε_{ij}^* through the series representation with analytical coefficients.

On the other hand stresses $\sigma_{\alpha\alpha}^0, \sigma_{\beta\beta}^0, \tau_{\alpha\beta}^0$ caused by the normal force that acts to the uniform half-space can be represented in the bipolar coordinates α and β Eq. (1a,b) with functions Φ_N^0 Eq. (9a) and relations of Eq. (2):

$$\sigma_{\alpha\alpha}^0 = \left(\frac{f_N}{a}\right) \frac{2 \sin^2 \frac{(2\beta-\beta_0)}{2} \sin^2 \frac{\beta_0}{2} \sinh^3 \alpha}{\pi (\cos\beta - \cosh\alpha)[\cos(\beta-\beta_0) - \cosh\alpha]^2}, \qquad (18a)$$

$$\sigma_{\beta\beta}^0 = \left(\frac{f_N}{a}\right) \frac{2 (\cos \frac{\beta_0}{2} - \cos \frac{2\beta-\beta_0}{2} \cosh\alpha)^2 \sin^2 \frac{\beta_0}{2} \sinh \alpha}{\pi (\cos\beta - \cosh\alpha)[\cos(\beta-\beta_0) - \cosh\alpha]^2}, \qquad (18b)$$

$$\tau_{\alpha\beta}^0 = \left(\frac{f_N}{a}\right) \frac{2 (\cos \frac{\beta_0}{2} - \cos \frac{2\beta-\beta_0}{2} \cosh\alpha) \sin \frac{2\beta-\beta_0}{2} \sin^2 \frac{\beta_0}{2} \sinh^2 \alpha}{\pi (\cos\beta - \cosh\alpha)[\cos(\beta-\beta_0) - \cosh\alpha]^2}. \qquad (18c)$$

Finally the total stresses can be found from Eq. (13) with the help of Eq. (2), Eq. (12), Eq. (17), and Eq. (18).

For completeness, we give a formula for the stress components $\sigma_{\beta\beta}$, arising at the boundary of the hole with the coordinate $\alpha = \alpha_1$:

$$\sigma_{\beta\beta}\big|_{\alpha=\alpha_1} = \sigma_{\beta\beta}^0\big|_{\alpha=\alpha_1} + \sigma_{\beta\beta}^*\big|_{\alpha=\alpha_1}, \qquad (19)$$

where

$$\sigma_{\beta\beta}^0\big|_{\alpha=\alpha_1} = \left(\frac{f_N}{a}\right) \frac{2(\cos \frac{\beta_0}{2} - \cos \frac{2\beta-\beta_0}{2} \cosh \alpha_1)^2 \sin^2 \frac{\beta_0}{2} \sinh \alpha_1}{\pi (\cos\beta - \cosh\alpha_1)[\cos(\beta-\beta_0) - \cosh\alpha_1]^2},$$

$$\sigma_{\beta\beta}^*\big|_{\alpha=\alpha_1} = -\left(\frac{f_N}{2\pi a}\right) \left\{ (1 - 2\cos 2\beta + \cosh 2\alpha_1) \operatorname{csch}^2 \alpha_1 \sin^2 \frac{\beta_0}{2} - 2 \csc \frac{\beta_0}{2} \sum_{n=2}^{\infty} \mathrm{H}_n(\alpha_1, \beta_0) \right\},$$

$$\mathrm{H}_n(\alpha_1, \beta_0) = e^{-n\alpha_1} (n \cos \tfrac{n\beta_0}{2} \sin \tfrac{\beta_0}{2} - \cos \tfrac{\beta_0}{2} \sin \tfrac{n\beta_0}{2}) \cdot$$

$$\cdot \Big(-\tfrac{2}{n} \cos \frac{n(2\beta-\beta_0)}{2} \sinh^2 \alpha_1 - \tfrac{2}{n(n^2-1)} (\cos\beta \cos \frac{n(2\beta-\beta_0)}{2} + n \sin\beta \sin \frac{n(2\beta-\beta_0)}{2})(\cosh\alpha_1 + n \sinh \alpha_1) +$$

$$+\frac{\cos\frac{n(2\beta-\beta_0)}{2}(\cos\beta-\cosh\alpha_1)}{2n(n^2\sinh^2\alpha_1-\sinh^2 n\alpha_1)}\cdot\Big\{\cosh\alpha_1(-2+n^2+2\cosh 2n\alpha_1)+$$

$$+n[-n\cosh 3\alpha_1+2\sinh\alpha_1(-1+\cosh 2n\alpha_1+2n^2\sinh^2\alpha_1+2\sinh 2n\alpha_1)]\Big\}\Big).$$

Stress component $\sigma_{\beta\beta}\big|_{\alpha=\alpha_1}$ as a function of the coordinate β is shown graphically in [8]. The behavior of stress component $\sigma_{\beta\beta}$ at the boundary of the hole coincides with results presented in [6].

We did not study the convergence of obtained series, representing the stresses σ^*_{ij}, in full details. However, we can note that for the parameters of the hole $\alpha_1<1$, the accuracy of the boundary conditions begins to fall in orders. Acceptable accuracy remains for the parameter $\alpha_1\approx 0.3$. For smaller parameters α_1 it is necessary to use special techniques to sum the series.

Biharmonic function Ψ^*_N caused by the presence of a hole is given by Eq. (15) and Eq. (16), taking into account the coefficients Eq. (17) and the following supplementary coefficients:

$$A_1^{add}=0, \tag{20a}$$

$$A_n^{add}=\frac{f_N(\lambda+2\mu)}{a\pi(\lambda+\mu)}\cdot$$

$$\cdot\frac{[n\sinh\alpha_1(\cosh\alpha_1+n\sinh\alpha_1)+e^{-n\alpha_1}\sinh n\alpha_1]}{n(n^2-1)(\sinh^2 n\alpha_1-n^2\sinh^2\alpha_1)}\sin\tfrac{n\beta_0}{2}(n\cos\tfrac{n\beta_0}{2}-\cot\tfrac{\beta_0}{2}\sin\tfrac{n\beta_0}{2}),\quad (n\geq 2) \tag{20b}$$

$$C_1^{add}=0, \tag{20c}$$

$$C_n^{add}=\frac{2f_N(\lambda+2\mu)}{a\pi(\lambda+\mu)}\cdot\frac{\sinh^2\alpha_1}{(\sinh^2 n\alpha_1-n^2\sinh^2\alpha_1)}\sin\tfrac{n\beta_0}{2}(n\cos\tfrac{n\beta_0}{2}-\cot\tfrac{\beta_0}{2}\sin\tfrac{n\beta_0}{2}),\quad (n\geq 2) \tag{20d}$$

$$D_1^{add}=-\frac{f_N(\lambda+4\mu)\coth\alpha_1\mathrm{csch}^2\alpha_1}{2a\pi(\lambda+\mu)}\sin^2\tfrac{\beta_0}{2}, \tag{20e}$$

$$D_n^{add}=-\frac{2f_N(\lambda+2\mu)}{a\pi(\lambda+\mu)}\cdot$$

$$\cdot\frac{[n\sinh\alpha_1(\cosh\alpha_1+n\sinh\alpha_1)+e^{-n\alpha_1}\sinh n\alpha_1]}{(n^2-1)(\sinh^2 n\alpha_1-n^2\sinh^2\alpha_1)}\cos\tfrac{n\beta_0}{2}(n\cos\tfrac{n\beta_0}{2}-\cot\tfrac{\beta_0}{2}\sin\tfrac{n\beta_0}{2}),\quad (n\geq 2). \tag{20f}$$

Under normal concentrate load the displacements of the straight boundary caused by a hole have a following simple form:

$$u^{*(N)}_{\alpha=0}(\beta)=u_1^{*(N)}(\beta)+\sum_{n=2}^{\infty}u_n^{*(N)}(\beta)\,, \tag{21a}$$

$$v^{*(N)}_{\alpha=0}(\beta)=v_1^{*(N)}(\beta)+\sum_{n=2}^{\infty}v_n^{*(N)}(\beta)\,, \tag{21b}$$

where

$$u_1^{*(N)}(\beta)=\frac{f_N(\lambda+2\mu)}{\pi\mu(\lambda+\mu)}\coth\alpha_1\operatorname{csch}^2\alpha_1\sin^2\tfrac{\beta}{2}\sin^2\tfrac{\beta_0}{2}\,,$$

$$u_n^{*(N)}(\beta)=\frac{f_N(\lambda+2\mu)}{\pi\mu(\lambda+\mu)}\cdot\frac{e^{-2n\alpha_1}\{-1+e^{2n\alpha_1}[1-n^2+n(n\cosh 2\alpha_1+\sinh 2\alpha_1)]\}}{n(n^2-1)(\sinh^2 n\alpha_1-n^2\sinh^2\alpha_1)}\cdot$$

$$\cdot\cos\tfrac{n(\beta-\beta_0)}{2}\csc\tfrac{\beta}{2}\csc\tfrac{\beta_0}{2}(n\cos\tfrac{n\beta}{2}\sin\tfrac{\beta}{2}-\cos\tfrac{\beta}{2}\sin\tfrac{n\beta}{2})(n\cos\tfrac{n\beta_0}{2}\sin\tfrac{\beta_0}{2}-\cos\tfrac{\beta_0}{2}\sin\tfrac{n\beta_0}{2}),\quad (n\ge 2)\,,$$

$$v_1^{*(N)}(\beta)=\frac{f_N(\lambda+2\mu)}{\pi\mu(\lambda+\mu)}\operatorname{csch}^2\alpha_1\sin^2\tfrac{\beta_0}{2}\sin\beta\,,$$

$$v_n^{*(N)}(\beta)=-\frac{f_N(\lambda+2\mu)}{\pi\mu(\lambda+\mu)}\cdot$$

$$v_n^{*(N)}(\beta)=-\frac{f_N(\lambda+2\mu)}{\pi\mu(\lambda+\mu)}\cdot\frac{2\sinh^2\alpha_1}{(\sinh^2 n\alpha_1-n^2\sinh^2\alpha_1)}(n\cos\tfrac{n\beta_0}{2}-\cot\tfrac{\beta_0}{2}\sin\tfrac{n\beta_0}{2})\cos\tfrac{n(\beta-\beta_0)}{2}\sin\tfrac{n\beta}{2},(n\ge 2)\,.$$

Expression Eq. (21a) shows that the displacements have the property, illustrating the principle of reciprocity of the work:

$$u^{*(N)}_{\alpha=0}(\beta=\beta_1,\beta_0=\beta_2)=u^{*(N)}_{\alpha=0}(\beta=\beta_2,\beta_0=\beta_1)\,. \tag{22}$$

2.5. Concentrated force with a magnitude of f_T is tangential to the planar boundary. The stress function Φ_T^* caused by the hole is represented by the Fourier series of Eq. (12) with the following coefficients:

$$B_0=f_T\left\{\frac{e^{-2\alpha_1}\sin\beta_0}{\pi(\cosh 2\alpha_1-1)}+\frac{2\sinh\alpha_1\sinh 2\alpha_1(\sinh\alpha_1-\cosh\alpha_1)\sin\beta_0}{\pi(\cosh 2\alpha_1-1)^2}\right\}, \tag{23a}$$

$$A_1^c=-f_T\,\frac{\sinh\alpha_1(\sinh\alpha_1-\cosh\alpha_1)\sin\beta_0}{\pi(\cosh 2\alpha_1-1)}\,, \tag{23b}$$

$$A_n^c=f_T\,\frac{[n\sinh\alpha_1(n\sinh\alpha_1-\cosh\alpha_1)+e^{-n\alpha_1}\sinh n\alpha_1]}{2\pi n(\sinh^2 n\alpha_1-n^2\sinh^2\alpha_1)}\sin n\beta_0,\quad (n\ge 2) \tag{23c}$$

$$C_n^c = -f_T \frac{(n-1)\sinh^2\alpha_1}{2\pi(\sinh^2 n\alpha_1 - n^2\sinh^2\alpha_1)}\sin n\beta_0, \quad (n \ge 2) \tag{23d}$$

$$A_1^s = -f_T \frac{e^{-2\alpha_1}\operatorname{csch}2\alpha_1(\cos\beta_0 - 1)}{2\pi}, \tag{23e}$$

$$A_n^s = A_n^c \tan\tfrac{n\beta_0}{2} = f_T \frac{[n\sinh\alpha_1(n\sinh\alpha_1 - \cosh\alpha_1) + e^{-n\alpha_1}\sinh n\alpha_1]}{\pi n(\sinh^2 n\alpha_1 - n^2\sinh^2\alpha_1)}\sin^2\tfrac{n\beta_0}{2}, \quad (n \ge 2), \tag{23f}$$

$$C_n^s = C_n^c \tan\tfrac{n\beta_0}{2} = -f_T \frac{(n-1)\sinh^2\alpha_1}{\pi(\sinh^2 n\alpha_1 - n^2\sinh^2\alpha_1)}\sin^2\tfrac{n\beta_0}{2}, \quad (n \ge 2). \tag{23g}$$

As a result the stress function Φ_T^*, given by expressions Eq. (12) and Eqs. (23) allows to find the stresses σ_{ij}^* and strains ε_{ij}^* through the series with analytic coefficients.

The stresses $\sigma_{\alpha\alpha}^0, \sigma_{\beta\beta}^0, \tau_{\alpha\beta}^0$ caused by the tangential force applied to the surface of the uniform half-space can be represented in the bipolar coordinates α and β Eqs. (1a,b) with the help of functions Φ_T^0 Eq. (9b) and relations Eqs. (2a,b,c):

$$\sigma_{\alpha\alpha}^0 = \left(\frac{f_T}{a}\right)\frac{2(-\cos\frac{2\beta-\beta_0}{2} + \cos\frac{\beta_0}{2}\cosh\alpha)\sin^2\frac{(2\beta-\beta_0)}{2}\sin\frac{\beta_0}{2}\sinh^2\alpha}{\pi(\cos\beta - \cosh\alpha)(\cos(\beta-\beta_0) - \cosh\alpha)^2}, \tag{24a}$$

$$\sigma_{\beta\beta}^0 = \left(\frac{f_T}{a}\right)\frac{2(\cos\frac{\beta_0}{2} - \cos\frac{2\beta-\beta_0}{2}\cosh\alpha)^2(-\cos\frac{2\beta-\beta_0}{2} + \cos\frac{\beta_0}{2}\cosh\alpha)\sin\frac{\beta_0}{2}}{\pi(\cos\beta - \cosh\alpha)(\cos(\beta-\beta_0) - \cosh\alpha)^2}, \tag{24b}$$

$$\tau_{\alpha\beta}^0 = \left(\frac{f_T}{a}\right)\cdot\left(\frac{2(\sin\frac{2\beta-3\beta_0}{2} + \sin\frac{6\beta-3\beta_0}{2} + 3\sin\frac{2\beta-\beta_0}{2} + \sin\frac{2\beta+\beta_0}{2})\sin\frac{\beta_0}{2}\sinh 2\alpha}{8\pi(\cos\beta - \cosh\alpha)(\cos(\beta-\beta_0) - \cosh\alpha)^2} - \right.$$

$$\left. - \frac{\sin(2\beta-\beta_0)\sin\beta_0(5\sinh\alpha + \sinh 3\alpha)]}{8\pi(\cos\beta - \cosh\alpha)(\cos(\beta-\beta_0) - \cosh\alpha)^2}\right). \tag{24c}$$

Finally the total stresses can be found from Eq. (13) with the help of Eq. (2), Eq. (12), Eq. (23), and Eq. (24).

Stress component $\sigma_{\beta\beta}\big|_{\alpha=\alpha_1}$ as a function of the coordinate β is shown graphically in [8].

From the analysis of equations and diagrams one can conclude that in the case of normal load stresses at the boundary of the hole increase significantly compared to how it would be in an uniform medium. There are areas of tension and compression. In the case of the tangential force effect of the hole is not so noticeable.

For the case of tangential load biharmonic function Ψ_T^* due to the presence of a hole is given by Eqs. (15) and (16), taking into account the coefficients Eq. (23) and the following supplementary coefficients:

$$A_1^{add} = 0, \tag{25a}$$

$$A_n^{add} = -\frac{f_T(\lambda+2\mu)}{a\pi(\lambda+\mu)} \cdot \frac{\sinh^2\alpha_1 \sin^2\frac{n\beta_0}{2}}{(\sinh^2 n\alpha_1 - n^2\sinh^2\alpha_1)}, \tag{25b}$$

$$C_1^{add} = \frac{f_T(2\lambda+5\mu)(\cos\beta_0 - 1)}{a\,\pi(\lambda+\mu)(e^{4\alpha_1}-1)}, \tag{25c}$$

$$C_n^{add} = -\frac{f_T(\lambda+2\mu)}{a\pi(\lambda+\mu)} \cdot \frac{2[n\sinh\alpha_1(-\cosh\alpha_1 + n\sinh\alpha_1) + e^{-n\alpha_1}\sinh n\alpha_1]}{n(\sinh^2 n\alpha_1 - n^2\sinh^2\alpha_1)}\sin^2\frac{n\beta_0}{2}, \quad (n\geq 2) \tag{25d}$$

$$D_1^{add} = -\frac{f_T(\lambda+4\mu)\operatorname{csch}^2\alpha_1 \sin\beta_0}{4a\pi(\lambda+\mu)}, \tag{25e}$$

$$D_n^{add} = \frac{f_T(\lambda+2\mu)}{a\pi(\lambda+\mu)} \cdot \frac{n\sinh^2\alpha_1 \sin n\beta_0}{(\sinh^2 n\alpha_1 - n^2\sinh^2\alpha_1)}, \quad (n\geq 2). \tag{25f}$$

Under tangential concentrate load the displacements at the boundary of the half-space caused by a hole have the following form:

$$u^{*(T)}_{\alpha=0}(\beta) = u_1^{*(T)}(\beta) + \sum_{n=2}^{\infty} u_n^{*(T)}(\beta), \tag{26a}$$

$$v^{*(T)}_{\alpha=0}(\beta) = v_1^{*(T)}(\beta) + \sum_{n=2}^{\infty} v_n^{*(T)}(\beta), \tag{26b}$$

where

$$u_1^{*(T)}(\beta) = \frac{f_T(\lambda+2\mu)}{2\pi\mu(\lambda+\mu)}\operatorname{csch}^2\alpha_1 \sin^2\frac{\beta}{2}\sin\beta_0,$$

$$u_n^{*(T)}(\beta) = -\frac{f_T(\lambda+2\mu)}{\pi\mu(\lambda+\mu)} \cdot \frac{2\sinh^2\alpha_1}{(\sinh^2 n\alpha_1 - n^2\sinh^2\alpha_1)}\left(n\cos\frac{n\beta}{2} - \cot\frac{\beta}{2}\sin\frac{n\beta}{2}\right)\cos\frac{n(\beta-\beta_0)}{2}\sin\frac{n\beta_0}{2},$$

$$(n\geq 2),$$

$$v_1^{*(T)}(\beta) = \frac{f_T(\lambda+2\mu)}{\pi\mu(\lambda+\mu)} \cdot \frac{[1+\cos\beta\cos\beta_0 - \cos\beta - \cos\beta_0 + (1+e^{2\alpha_1})\sin\beta\sin\beta_0]}{(-1+e^{4\alpha_1})},$$

$$v_n^{*(T)}(\beta) = \frac{f_T(\lambda+2\mu)}{16\pi\mu(\lambda+\mu)} \cdot \frac{(-2e^{-2n\alpha_1} + e^{2\alpha_1}(n-1)n + e^{-2\alpha_1}n(n+1) - 2(n^2-1))}{n(\sinh^2 n\alpha_1 - n^2\sinh^2\alpha_1)} \cdot$$

$$(2\sin n\beta_0\sin n\beta + 2 - 2\cos n\beta - 2\cos n\beta_0 + \cos n(\beta-\beta_0) + \cos n(\beta+\beta_0)), \quad (n\geq 2).$$

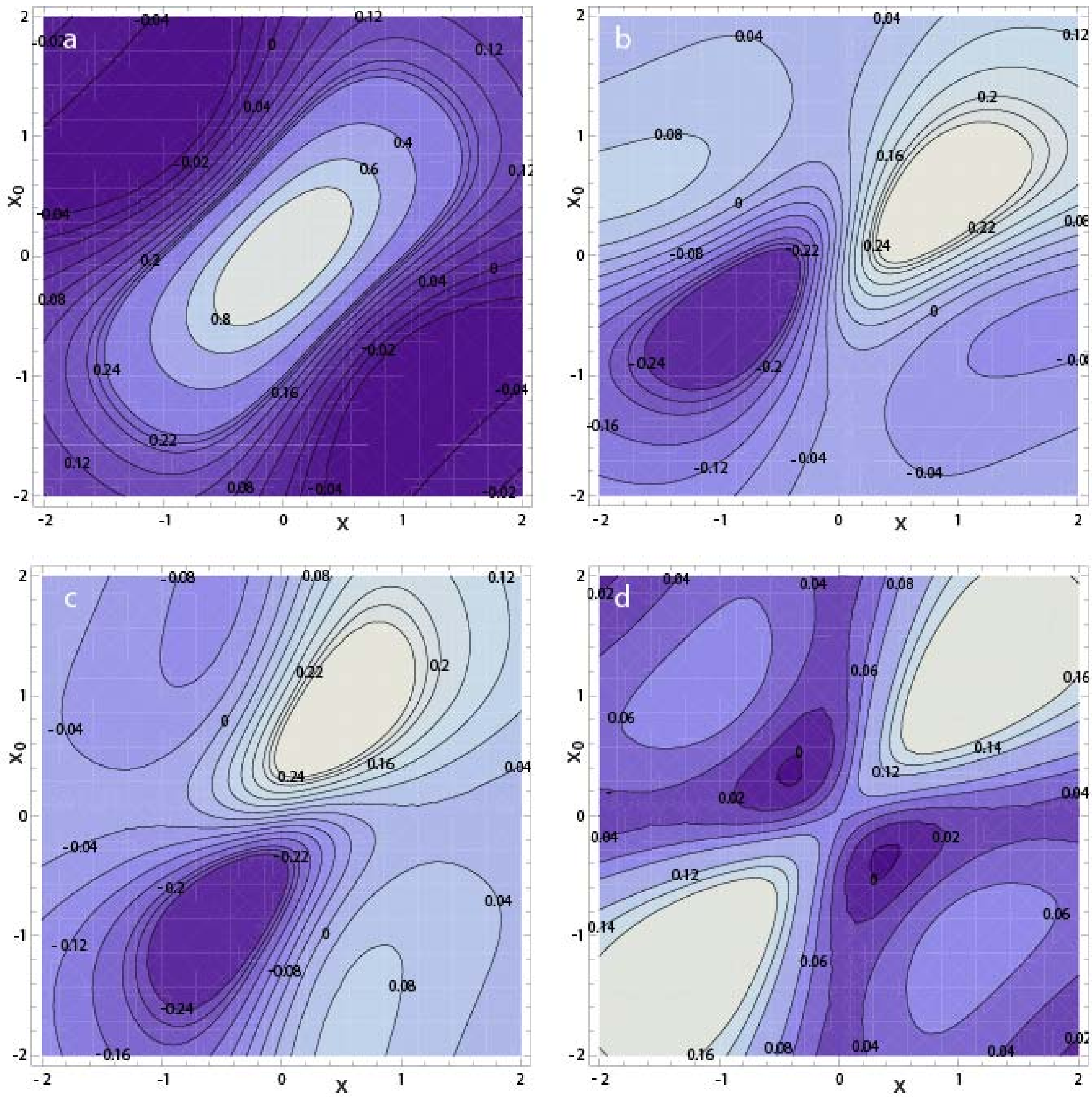

Fig. 3. Contour maps of the surface displacements due to hole under the action of the concentrated force. (a) normal surface displacements $u^{*(N)}_{\alpha=0}$ at normal force; (b) the tangential surface displacements $v^{*(N)}_{\alpha=0}$ at normal force; (c) normal surface displacements $u^{*(T)}_{\alpha=0}$ at tangential force; (d) the tangential surface displacements $v^{*(T)}_{\alpha=0}$ at tangential force. Coordinate x is a coordinate of the measured displacement; x_0 is a coordinate of applied force. The linear quantities are expressed in units of the radius of the hole ρ. The displacements $u^{*(N)}_{\alpha=0}$ and $v^{*(N)}_{\alpha=0}$ are presented in units $\frac{f_N(\lambda+2\mu)}{\mu(\lambda+\mu)}$, where f_N is a value of the applied normal force, λ, μ are the elastic modules. The displacements $u^{*(T)}_{\alpha=0}$ and $v^{*(T)}_{\alpha=0}$ are presented in units $\frac{f_T(\lambda+2\mu)}{\mu(\lambda+\mu)}$, where f_T is a value of the applied tangential force. Distance between center of the hole and planar boundary is $d = 1.43\rho$.

Expression Eq. (26b) shows that the displacements have the following property:

$$v_{\alpha=0}^{*(T)}(\beta=\beta_1,\beta_0=\beta_2)=v_{\alpha=0}^{*(T)}(\beta=\beta_2,\beta_0=\beta_1)\,. \tag{27}$$

Comparison of $u_{\alpha=0}^{*(T)}$ and $v_{\alpha=0}^{*(N)}$ demonstrates the expected correlation:

$$v_{\alpha=0}^{*(N)}(\beta=\beta_0'',\beta_0')f_T(\beta_0'')=u_{\alpha=0}^{*(T)}(\beta=\beta_0',\beta_0'')f_N(\beta_0') \tag{28}$$

Figure 3 shows the planar surface displacements due to the hole.

It is becomes obvious that the hole has the greatest impact on the normal displacement of planar boundary under the action of the normal force.

3. Inverse problem: Finding the parameters of the hole from the measured surface displacements

Let us assume now that there is a possibility to measure the surface displacements. One can measure (i) the surface displacements for the body with a defect (presumably hole) loaded by concentrated linear force and (ii) the surface displacements for the solid body without a hole loaded by concentrated linear force. For the surface displacements of loaded hole-less half-space the exact formulas can be used also [7]. In other words the displacements $u_{\alpha=0}^{(N)}, v_{\alpha=0}^{(N)}, u_{\alpha=0}^{(T)}, v_{\alpha=0}^{(T)}$ for the body with a hole and $u_{\alpha=0}^{0(N)}, v_{\alpha=0}^{0(N)}, u_{\alpha=0}^{0(T)}, v_{\alpha=0}^{0(T)}$ for the standard body without a hole are assumed to be known. Coordinates of points of force applications and the response points (i.e. the places where the displacements were measured) are the same for the body with hole and for the standard body. Hence we possess the information on the additional displacements $u_{\alpha=0}^{*(N)}, v_{\alpha=0}^{*(N)}, u_{\alpha=0}^{*(T)}, v_{\alpha=0}^{*(T)}$ due to the hole in the tested body. Note that for the plane elasticity problem, instead of the points we mean the lines.

We are interested in a possibility to determine the parameters of the hole, namely the hole size ρ and the hole depth h (Fig. 2) on the base of measured displacement maps $u_{\alpha=0}^{*(N)}, v_{\alpha=0}^{*(N)}, u_{\alpha=0}^{*(T)}, v_{\alpha=0}^{*(T)}$, similar to the maps shown in Fig. 3.

The algorithm will include a number of steps. From the maps one can easily determine the place on the surface, under which a hole is localized, see Fig. 3. This will serve as the origin of Cartesian coordinate system associated with a hole, see Fig. 2.

Then the distance $|x|$ measured along the surface between the origin and the response point β should be related to bipolar coordinate β and scale parameter a by equation:

$$|x|=a\,|\cot\tfrac{\beta}{2}|; \tag{29}$$

If the force is applied at the origin of the Cartesian coordinate x=0, the bipolar coordinate for the force β_0 is equal to $=\pm\pi$ in formulas for surface displacements Eq. (21) and Eq. (26). Then bipolar coordinate of the hole α_1 and bipolar coordinate of the response point β can be extracted from two equations, for instance Eq. (21a) and Eq. (21b) or from Eq. (21a) and Eq. (26b) on the base of the measured displacements, the known magnitude of the force and the known elastic modules $\mu,\quad\lambda$.

The above said can be illustrated by particular examples shown in Table 1.

From Table 1 we obtain the hole radius ρ, the hole depth h, the coordinate of the center of the hole d, and the aspect ratio of the hole $\delta = d/\rho = \cosh\alpha_1$ on the base of the measured distance $|x|$ between the origin and the response point β and measured surface displacements. Note that origin is easy to find on the basis of map (Fig. 3a), because in this case the displacements $u^{*(N)}_{\alpha=0}$ have maximum in the origin. In Table 1 the magnitudes of calculated parameters α_1, β are in good agreement with parameters α_1, β which were put initially into Eq. (21a) and Eq. (26b) to get so-called measured surface displacements.

Of course, finding the roots of equations Eq. (21a) and Eq. (26b) at the given left sides requires knowledge of the areas where the roots can be found. But fortunately wrong areas of finding the roots, lead to solutions that go beyond the permissible parameters α_1, β: $\alpha_1 > 0$, $-\pi \le \beta \le \pi$. This causes us to look for the roots of equations in other areas. It should also be noted that the accuracy of the roots depends on the accuracy of the given left sides in equations.

Table 1. Finding of hole parameters from measured surface displacements.

Measured distance $\|x\|$ between the origin and the response point β, in arbitrary units	Measured surface displacements caused by force applied at point $\beta_0 = \pi$ *)	Bipolar coordinates, calculated from Eq. (21) and Eq. (26) on the base of measured surface displacements	Scale parameter a calculated from Eq. (29): $a = \|x\| / \|\cot\frac{\beta}{2}\|$, in the same units as measured distance $\|x\|$	Radius of the hole $\rho = a/\sinh\alpha_1$, hole depth $h = a\tanh\frac{\alpha_1}{2}$, the coordinate of the center of the hole $d = a\coth\alpha_1$, in the same units as measured distance $\|x\|$; aspect ratio $\delta = d/\rho = \cosh\alpha_1$
1	$u^{*(N)}_{\alpha=0}(\alpha_1,\beta) \approx 0.115$ $v^{*(T)}_{\alpha=0}(\alpha_1,\beta) \approx 0.014$	$\alpha_1 \approx 1.1$, $\beta \approx 1.57 \approx \pi/2$	~1	$\rho \approx 0.75$ $h \approx 0.50$ $d \approx 1.25$ $\delta \approx 1.67$
1	$u^{*(N)}_{\alpha=0}(\alpha_1,\beta) \approx 0.114$ $v^{*(T)}_{\alpha=0}(\alpha_1,\beta) \approx 0.0024$	$\alpha_1 \approx 1.51$, $\beta \approx 2.12 \approx 2\pi/3$	~1.78	$\rho \approx 0.83$ $h \approx 1.14$ $d \approx 1.96$ $\delta \approx 2.37$

*) The units of displacements $u^{*(N)}$, $v^{*(N)}$ are $\frac{f_N(\lambda+2\mu)}{\pi\mu(\lambda+\mu)}$, for displacements $u^{*(T)}$, $v^{*(T)}$ the units of measurement are $\frac{f_T(\lambda+2\mu)}{\pi\mu(\lambda+\mu)}$.

4. Conclusions

We have presented the analysis of a problem on the identification of defects in a solid body with the help of surface displacements.

As a good example, the plane problem on finding the elastic fields in a half-space weakened by a circular hole and loaded by a concentrated force was considered. We have presented an analytical solution to this problem. The solution operates with biharmonic functions that allow to calculate all elastic fields, in particular, the planar surface displacements, which are modified by the presence of a hole. Biharmonic functions are given in terms of Fourier series. The surface displacements are presented in the analytical form of series with compact coefficients.

It is demonstrated that the hole size and the depth of hole buried under the surface are calculated on the base of analytical formulas for the surface displacements.

We expect that the examination of surface displacements caused by an applied force will be used for the determination of the parameters of the *inhomogeneities, voids* and *cracks* placed in the near-surface layers of elastic bodies. The described method can be applied for the objects (defects) that modify the elastic fields caused by the applied force, i.e. for defects possessing the *inhomogeneity* property. This analysis is not suitable for *dislocations, disclinations* and *pure elastic inclusions* (inclusions without inhomogeneity e.g. dilatation centers). The above mentioned defects are the sources of their own elastic fields including the surface displacements. The investigation of characteristic surface displacement patterns for these defects has to be performed separately and should also address the presence of plastic component of displacement as it, for example, has been done in the study of typical "cross-hatch" surface profile [9] related to the formation of misfit dislocations and observed in relaxed lattice mismatched layers.

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