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UNIQUENESS, RECIPROCITY THEOREM AND VARIATIONAL PRINCIPLE IN FRACTIONAL ORDER THEORY OF THERMOELASTICITY WITH VOIDS

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Abstract. In this work, a new theory of thermoelasticity with voids is discussed by using the methodology of fractional calculus. The governing equations for particle motion in a homogeneous anisotropic fractional order thermoelastic medium with voids are presented. A variational principle, uniqueness theorem and reciprocity theorem are proved. The plane wave propagation in orthotropic thermoelastic material with fractional order derivative and voids is studied. For two-dimensional problem there exist quasi-longitudinal (qP) wave, quasi-transverse (qS) wave, quasi-longitudinal thermal (qT) wave and a quasi-longitudinal volume fractional (qV) wave. From the obtained results the different characteristics of waves like phase velocity, attenuation coefficient, specific loss and penetration depth are computed numerically and presented graphically.

1. Introduction

The study of dynamic properties of elastic solids is significant in the ultrasonic inspection of materials, vibrations of engineering structures, in seismology, geophysical and various other fields. Such materials are usually described by equations of linear elastic solids, however there are materials of a more complex microstructure (composite materials, granular materials, soils etc.) depict specific characteristic response to applied load.

Theory of linear elastic materials with voids is one of the most important generalizations of the classical theory of elasticity. This theory has practical utility for investigating various types of geological and biological materials for which elastic theory is inadequate. This theory is concerned with elastic materials consisting of a distribution of small pores (voids), in which the void volume is included among the kinematics variables and in the limiting case of volume tending to zero, the theory reduces to the classical theory of elasticity.

A nonlinear theory of elastic materials with voids was developed by Nunziato and Cowin [1]. Later, Cowin and Nunziato [2] developed a theory of linear elastic materials with voids for the mathematical study of the mechanical behavior of porous solids. They considered several applications of the linear theory by investigating the response of the materials to homogeneous deformations, pure bending of beams and small amplitudes of acoustic waves. The small acoustic waves in an infinite elastic medium with voids showed that two distinct types of longitudinal waves and a transverse wave can propagate without affecting the porosity of the material and without attenuation. The two types of longitudinal waves are attenuated and dispersed; one longitudinal wave is associated with elastic property of the material and the second associated with the property of the change in porosity of the material. These longitudinal acoustic waves are both attenuated and dispersed due to the change in the material porosity.

Singh and Tomar [3] investigated the propagation of plane waves in an infinite thermoelastic medium with voids. They found that three coupled longitudinal waves and one transverse waves can exist in an infinite thermoelastic medium with voids. Singh [4] studied the propagation of plane waves in a generalized thermoelastic material with voids in the context of Lord-Shulman [5] theory. He showed that there exist three compressional and one shear wave in an infinite generalized thermoelastic material with voids. Ciarletta and Straughan [6] presented a theory for acceleration wave propagation in an elastic material with voids, which allows heat travel at infinite speed.

During recent years, several interesting models have been developed by using fractional calculus to study the physical processes particularly in the area of heat conduction, diffusion, viscoelasticity, mechanics of solids, control theory, electricity etc. It has been realized that the use of fractional order derivatives and integrals leads to the formulation of certain physical problems which is more economical and useful than the classical approach. There exists many material and physical situations like amorphous media, colloids, glassy and porous materials, manmade and biological materials/polymers, transient loading etc., where the classical thermoelasticity based on Fourier type heat conduction breaks down. In such cases, one needs to use a generalized thermoelasticity theory based on an anomalous heat conduction model involving time fractional (non integer order) derivatives.

Povstenko [7] proposed a quasi-static uncoupled theory of thermoelasticity based on the heat conduction equation with a time-fractional derivative of order α. Because the heat conduction equation in the case $1 \leq \alpha \leq 2$ interpolates the parabolic equation $\alpha = 1$ and the wave equation $\alpha = 2$, this theory interpolates a classical thermoelasticity and a thermoelasticity without energy dissipation. He also obtained the stresses corresponding to the fundamental solutions of a Cauchy problem for the fractional heat conduction equation for one-dimensional and two-dimensional cases.

Povstenko [8] investigated the nonlocal generalizations of the Fourier law and heat conduction by using time and space fractional derivatives. Youssef [9] proposed a new model of thermoelasticity theory in the context of a new consideration of heat conduction with fractional order and proved the uniqueness theorem. Jiang and Xu [10] obtained a fractional heat conduction equation with a time fractional derivative in the general orthogonal curvilinear coordinate and also in other orthogonal coordinate system. Povstenko [11] investigated the fractional radial heat conduction in an infinite medium with a cylindrical cavity and associated thermal stresses.

Ezzat [12] constructed a new model of the magneto-thermoelasticity theory in the context of a new consideration of heat conduction equation by using the Taylor series expansion of time fractional order, developed by Jumarie [13] as

$$\left[q + \frac{\tau_0^\alpha}{\alpha!} \frac{\partial^\alpha q}{\partial t^\alpha} \right] = -K \nabla T \,.$$

Ezzat [14] studied the problem of state space approach to thermoelectric fluid with fractional order heat transfer. The Laplace transform and state-space techniques were used to solve a one-dimensional application for a conducting half space of thermoelectric elastic material. Povstenko [15] investigated the generalized Cattaneo-type equations with time fractional derivatives and formulated the theory of thermal stresses. Biswas and Sen [16] proposed a scheme for optimal control and a pseudo state space representation for a particular type of fractional dynamical equation.

Uniqueness, reciprocity theorems and variational principle in the theory of thermoelastic materials with voids were established by Iesan [17]. Uniqueness and reciprocity theorems for the equations of generalized thermoleastic diffusion problem, in isotropic media, was proved by Sherief et al. [18] on the basis of variational principle equation. Aouadi [19, 20] proved the uniqueness and reciprocity theorems for the equations of generalized thermoleastic diffusion problem in both isotropic and anisotropic media by using the Laplace transformation method. Sherief et al. [21] introduced a new model of thermoelasticity using fractional calculus, proved a uniqueness theorem, and derived a reciprocity relation and a variational principle. El-Karamany and Ezzat [22] introduced two models where the fractional derivatives and integrals are used to modify the Cattaneo heat conduction law [23] and in the context of two temperature thermoelasticity theory, uniqueness and reciprocity theorems are proved, the convolution principle is given and is used to prove a uniqueness theorem with no restrictions imposed on the elasticity or thermal conductivity tensors except symmetry conditions. Rusu [24] studied the existence and uniqueness in thermoelastic materials with voids. Marin [25] presented the contributions on uniqueness in thermoelastodynamics for bodies with voids. Puri and Cowin [26] studied the behavior of plane waves in linear elastic materials with voids. Domain of influence theorem in the linear theory of elastic materials with voids was discussed by Dhaliwal and Wang [27]. Birsan [28] established existence and uniqueness of weak solution in the linear theory of elastic shells with voids.

Sharma [29] studied the existence of longitudinal and transverse waves in anisotropic thermoelastic media. Analysis of wave motion at the boundary surface of orthotropic thermoelastic material with voids and isotropic elastic half-space was studied by Kumar and Kumar [30].

The current manuscript is an attempt to combine these previous results with the fractional order theory of generalized thermoelasticity with void. The variational principle is proved and is used to prove the uniqueness theorem. Reciprocity theorem is also proved. Plane wave propagation in orthotropic thermoelastic medium is studied. Some special cases are also deduced.

2. Basic equations

Following Iesan [17], the basic governing equations in anisotropic, homogeneous thermoelastic solids with fractional order derivative and voids are:

Constitutive Relations:

$$\sigma_{ij} = c_{ijlm}e_{lm} + B_{ij}\phi + D_{ijk}\sigma_{,k} - \beta_{ij}T, \tag{1}$$

$$h_i = A_{ij}\phi_{,j} + D_{mni}e_{mn} + d_i\phi - a_iT, \tag{2}$$

$$g = -B_{ij}e_{,ij} - \varsigma\phi - d_i\phi_{,i} + mT, \tag{3}$$

$$\rho\eta = \beta_{ij}e_{,ij} + m\phi + a_i\phi_{,i} + aT, \tag{4}$$

$$q_i = -K_{ij}T_{,j}, \tag{5}$$

Equations of motion:

$$\sigma_{ij,j} + \rho F_i = \rho\ddot{u}_i, \tag{6}$$

Equations of equilibrated forces:

$$h_{i,i} + g + \rho l = \rho\chi\ddot{\phi}, \tag{7}$$

Here, $\phi = \psi - \psi_0$, $T=\theta-T_0$ is the temperature change, θ is the absolute temperature, T_0 is the reference uniform temperature of the body chosen such that $|T / T_0| << 1$, ρ is the mass density, q_i is the heat conduction vector, χ is the equilibrated inertia, ψ is the volume distribution function which in the reference state is ψ_0, ϕ is the volume fraction field, h_i are the components of the equilibrated stress vector, g is the intrinsic equilibrated force, l is the extrinsic equilibrated force, $K_{ij} = K_{ji}$ is the thermal conductivity tensor, $c_{ijkm}\left(c_{ijkm} = c_{kmij} = c_{jikm} = c_{ijmk}\right)$ is the tensor of elastic constants, $\sigma_{ij} = \sigma_{ji}$ are the components of the stress tensor, u_i are the components of the displacement vector, $e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$, are the components of the strain tensor, η is the entropy per unit mass, $d_i, a_i, \varsigma, m, a, A_{ij}\left(= A_{ji}\right), \beta_{ij}\left(= \beta_{ji}\right), D_{ijk}\left(= D_{jik}\right)$, and $B_{ij} = B_{ji}$ are the characteristic functions of the material.

Fractional integral and derivative. The Riemann-Liouville fractional integral is introduced as a natural generalization of the convolution type integral (Miller and Ross [31], Podlubny [32], Povstenko [7], Povstenko [8]):

$$I^{\alpha}f(t) = \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)d\tau, \quad \alpha > 0. \tag{8}$$

The Laplace transform rule for this fractional integral is

$$L\left[I^{\alpha}f(t)\right] = \frac{1}{s^{\alpha}}L\left[f(t)\right]. \tag{9}$$

The Riemann-Liouville fractional derivative of fractional order α is defined as the left-inverse of the fractional integral I^{α} as

$$D_{RL}^{\alpha}f(t) = D^{n}I^{n-\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1}f(\tau)d\tau, \quad n-1 < \alpha < n. \tag{10}$$

and for Laplace transform, the initial values of the fractional integral $I^{n-\alpha}f(t)$ and the derivatives of its of order k=1, 2, 3,..., n-1 are required, where

$$L\left[D_{RL}^{\alpha}f(t)\right] = s^{\alpha}L\left[f(t)\right] - \sum_{k=0}^{n-1} s^{n-1-k}D^{k}I^{n-\alpha}f(0), \quad n-1 < \alpha < n. \tag{11}$$

Following Ezzat [16], the Fourier law for Lord-Shulman (LS) model with fractional order derivative is

$$q_i\left[1 + \frac{\tau_0^{\alpha}}{\alpha!}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right] = -K_{ij}T_{,j}. \tag{12}$$

The heat conduction equation for Lord-Shulman (LS) model with fractional order derivative in absence of heat sources is

$$\rho C^{*}\frac{\partial}{\partial t}\left[T+\frac{\tau_0^{\alpha}}{\alpha!}\frac{\partial^{\alpha}T}{\partial t^{\alpha}}\right]+T_0\left[1+\frac{\tau_0^{\alpha}}{\alpha!}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right]\left(\beta_{ij}\dot{u}_{i,j}+m\dot{\phi}\right)=K_{ij}T_{,ij}, \tag{13}$$

where τ_0 is the thermal relaxation time, which will ensure that the heat conduction equation will predict finite speeds of heat propagation, C^{*} is the specific heat at the constant strain, α denotes the fractional order.

If the material symmetry is of a type that posses a center of symmetry then D_{ijk}, d_i, a_i are identically zero. Therefore the system of equations (1)-(4) becomes:

$$\sigma_{ij}=c_{ijlm}e_{lm}+B_{ij}\phi-\beta_{ij}T, \tag{14}$$

$$h_i=A_{ij}\phi_{,j}, \tag{15}$$

$$g=-B_{ij}e_{,ij}-\varsigma\phi+mT, \tag{16}$$

$$\rho\eta=\beta_{ij}e_{,ij}+m\phi+aT. \tag{17}$$

3. Variational principle

The principle of virtual work with variation of displacements for elastic deformable body with voids is:

$$\int_V\left[\rho\left(f_i-\ddot{u}_i\right)\delta u_i+\rho\left(l-\chi\ddot{\phi}+g\right)\delta\phi\right]dV+\int_A\left[T_i\delta u_i+h\delta\phi\right]dA=\int_V\left[\sigma_{ij}\delta u_{i,j}+h_i\delta\phi_{,i}\right]dV. \tag{18}$$

On the left hand side, we have the virtual work of body forces f_i, l, g, inertial forces $\rho\ddot{u}_i, \rho\chi\ddot{\phi}$, surface forces $T_i=\sigma_{ij}n_j, h=h_in_i$ whereas on the right hand side, we have the virtual work of internal forces. We denote by n_j the outward unit normal of ∂V.

Using the symmetry of stress tensor and the definition of the strain tensor, the equation (18) can be written in the alternative form as

$$\int_V\left[\rho\left(f_i-\ddot{u}_i\right)\delta u_i+\rho\left(l-\chi\ddot{\phi}+g\right)\delta\phi\right]dV+\int_A\left[T_i\delta u_i+h\delta\phi\right]dA=\int_V\left[\sigma_{ij}\delta e_{ij}+h_i\delta\phi_{,i}\right]dV. \tag{19}$$

Substituting the value of σ_{ij} from relation (1) and h_i from relation (2) in equation (19), we obtain

$$\int_V\left[\rho\left(f_i-\ddot{u}_i\right)\delta u_i+\rho\left(l-\chi\ddot{\phi}+g\right)\delta\phi\right]dV+\int_A\left[T_i\delta u_i+h\delta\phi\right]dA=$$

$$=\delta W+\delta E+\int_V B_{ij}\phi\delta e_{ij}dV-\int_V\beta_{ij}T\delta e_{ij}dV, \tag{20}$$

where

$$W=\frac{1}{2}\int_V c_{ijlm}e_{ij}e_{lm}dV, E=\frac{1}{2}\int_V A_{ij}\phi_{,i}^2dV. \tag{21}$$

We define a vector **J** (Biot [33]) connected with the entropy through the relation

$$\rho\eta = -J_{i,i}. \tag{22}$$

Using Eqs.(13), (17) and (22), we obtain

$$T_0 L_{ij}\left(\frac{d}{dt} + \frac{\tau_0^\alpha}{\alpha!}\frac{d^{\alpha+1}}{dt^{\alpha+1}}\right)J_i + T_{,j} = 0, \tag{23}$$

$$-J_{i,i} = \beta_{ij}e_{,ij} + m\phi + aT, \tag{24}$$

where L_{ij} is the resistivity matrix and is the inverse of the thermal conductivity K_{ij} and we used the notation $\rho C^* = aT_0$.

Multiplying both sides of Eq. (23) by δJ_i and integrating over the region of the body and using the divergence theorem with the aid of (24), we obtain

$$\int_A (T\delta J_j)n_j dA + \int_V \beta_{ij}T\delta e_{ij}dV + m\int_V T\delta\phi dV + \delta(F+H) = 0, \tag{25}$$

where

$$F = \frac{a}{2}\int_V T^2 dV, F = a\int_V T\delta T dV, \tag{26}$$

$$H = \frac{T_0}{2}\int_V L_{ij}\left(\frac{dJ_i}{dt} + \frac{\tau_0^\alpha}{\alpha!}\frac{d^{\alpha+1}J_i}{dt^{\alpha+1}}\right)J_j dV, \delta H = T_0\int_V L_{ij}\left(\frac{dJ_i}{dt} + \frac{\tau_0^\alpha}{\alpha!}\frac{d^{\alpha+1}J_i}{dt^{\alpha+1}}\right)\delta J_j dV, \tag{27}$$

Substituting the value of g from (3) in relation (20), we obtain

$$\int_V \left[\rho(f_i - \ddot{u}_i)\delta u_i + \rho(l - \chi\ddot{\phi})\delta\phi\right]dV + \int_A [T_i\delta u_i + h\delta\phi]dA + m\int_V T\delta\phi dV + \int_V \beta_{ij}T\delta e_{ij}dV =$$

$$= \delta(W + E + R + G), \tag{28}$$

where

$$R = \int_V B_{ij}e_{ij}\phi dV, G = \frac{\varsigma}{2}\int_V \phi^2 dV. \tag{29}$$

Eliminating integrals $\int_V \beta_{ij}T\delta e_{ij}dV$ and $m\int_V T\delta\phi dV$ from Eqs. (25) and (28), we obtain the variational principle in the following form:

$$\delta(W + E + R + G + H + F) = \int_V \rho(f_i - \ddot{u}_i)\delta u_i dV + \int_V \rho(l - \chi\ddot{\phi})\delta\phi dV +$$

$$+\int_A T_i\delta u_i dA + \int_A h\delta\phi dA - \int_A T\delta J_i n_i dA. \tag{30}$$

On the right-hand side of Eq. (30), we find all the causes, the mass forces, inertial forces, the surface forces, the heating potential and equilibrated stress vector on the surface A bounding the body.

4. Uniqueness theorem

We assume that the virtual displacements δu_i, the virtual increment of the temperature δT, etc. correspond to the increments occurring in the body. Then

$$\delta u_i = \frac{\partial u_i}{\partial t}dt = \dot{u}_i dt, \delta T = \frac{\partial T}{\partial t}dt = \dot{T}dt, \quad \text{etc} \tag{31}$$

and Eq. (30) with the aid of (31) reduces to the following form:

$$\delta(W + E + R + G + H + F) = \int_V \rho(f_i - \ddot{u}_i)\dot{u}_i dV + \int_V \rho\left(l - \chi\ddot{\phi}\right)\dot{\phi}dV +$$

$$+\int_A T_i\dot{u}_i dA + \int_A h\dot{\phi}dA - \int_A T\dot{J}_i n_i dA. \tag{32}$$

Now

$$\int_V \rho\ddot{u}_i\dot{u}_i dV = \frac{\partial \aleph}{\partial t}, \tag{33}$$

$$\int_V \rho\chi\ddot{\phi}\dot{\phi}dV = \frac{\partial D}{\partial t}, \tag{34}$$

where $\aleph = \frac{1}{2}\int_V \rho\dot{u}_i\dot{u}_i dV$ and $D = \int_V \rho\chi\dot{\phi}\dot{\phi}dV$ are the kinetic energy of the body enclosed by the volume V.
We also have

$$F + E + G = \frac{1}{2}\int_V \left(T^2 + A_{ij}\phi_{,i}^2 + \varsigma\phi^2\right)dV, \tag{35}$$

Making use of (33), (34) and (35) in Eq. (32), we obtain

$$\frac{d}{dt}\left(W + R + H + \aleph + D + \frac{1}{2}\int_V \left(T^2 + A_{ij}\phi_{,i}^2 + \varsigma\phi^2\right)dV\right)dV =$$

$$= \int_V \rho f_i\dot{u}_i dV + \int_V \rho l\dot{\phi}dV + \int_A T_i\dot{u}_i dA + \int_A h\dot{\phi}dA - \int_A T\dot{J}_i n_i dA. \tag{36}$$

The above equation is the basis for the proof of the following uniqueness theorem.

Theorem. There is only one solution of the equations (6), (7) and (13), subject to boundary conditions on the surface A

$$T_i = \sigma_{ij}n_j = \overline{T}_i, \quad h = h_i n_j = \overline{h}, \quad u_i = \overline{u}_i, \quad T = \overline{T}, \quad \phi = \overline{\phi},$$

and the initial conditions at t=0

$$u_i = u_i^0, \quad \dot{u}_i = \dot{u}_i^0, \quad T = T^0, \quad \dot{T} = \dot{T}^0, \quad \phi = \phi^0, \quad \dot{\phi} = \dot{\phi}^0,$$

where $h_{i1}, \overline{T}, \overline{\phi}, u_i^0, \dot{u}_i^0, T^0, \dot{T}^0, \phi^0, \dot{\phi}^0$ are known functions.

We assume that the material parameters satisfy the inequalities

$$T_0 > 0, \quad \tau_0 > 0, \quad C_E > 0, \quad \rho > 0, \quad \varsigma > 0, \quad \chi > 0, \quad m > 0, \quad a > 0 \tag{37}$$

and $c_{ijlm}, L_{ij}, A_{ij}, B_{ij}, \beta_{ij}, D_{ijk}$ are positive definite.

Proof. Let $u_i^{(1)}, T^{(1)}, \phi^{(1)}, ...$ and $u_i^{(2)}, T^{(2)}, \phi^{(2)}, ...$ be two solution sets of equations (6), (7), (12)-(17). Let us take

$$u_i = u_i^{(1)} - u_i^{(2)}, T = T^{(1)} - T^{(2)}, \phi = \phi^{(1)} - \phi^{(2)}. \tag{38}$$

The functions u_i, T and ϕ satisfy the governing equations with zero body forces and homogeneous initial and boundary conditions. Thus these functions satisfy an equation similar to the equation (36) with zero right hand side, that is

$$\frac{d}{dt}\left(W + R + H + \aleph + D + \frac{1}{2}\int_V \left(T^2 + A_{ij}\phi_{,i}^2 + \varsigma\phi^2\right)dV\right)dV = 0, \tag{39}$$

Since we have $L_{ij} = L_{ji}$, from equation (27), we obtain

$$\frac{dH}{dt} = T_0\int_V L_{ij}\dot{J}_i\dot{J}_j dV + \frac{d}{dt}\left[\frac{T_0}{2}\frac{\tau_0^\alpha}{\alpha!}\int_V L_{ij}\dot{J}_i\frac{d^{\alpha+1}J_i}{dt^{\alpha+1}}\right]dV, \tag{40}$$

Using (40) in (39), we obtain

$$\frac{d}{dt}\left(W + R + \aleph + D + \frac{1}{2}\int_V \left(T^2 + A_{ij}\phi_{,i}^2 + \varsigma\phi^2\right)dV + \frac{T_0}{2}\frac{\tau_0^\alpha}{\alpha!}\int_V L_{ij}\dot{J}_i\frac{d^{\alpha+1}J_i}{dt^{\alpha+1}}\right)dV + T_0\int_V L_{ij}\dot{J}_i\dot{J}_j dV = 0. \tag{41}$$

Using inequalities (37) in (41), we obtain

$$\frac{d}{dt}\left(W + R + \aleph + D + \frac{1}{2}\int_V \left(T^2 + A_{ij}\phi_{,i}^2 + \varsigma\phi^2\right)dV + \frac{T_0}{2}\frac{\tau_0^\alpha}{\alpha!}\int_V L_{ij}\dot{J}_i\frac{d^{\alpha+1}J_i}{dt^{\alpha+1}}dV\right) \le 0. \tag{42}$$

We thus see that the expression

$$W + R + \aleph + D + \frac{1}{2}\int_V \left(T^2 + A_{ij}\phi_{,i}^2 + \varsigma\phi^2\right)dV + \frac{T_0}{2}\frac{\tau_0^\alpha}{\alpha!}\int_V L_{ij}\dot{J}_i\frac{d^{\alpha+1}J_i}{dt^{\alpha+1}}dV \tag{43}$$

is a decreasing function of time. We also note that the expression $\frac{1}{2}\int_V \left(T^2 + A_{ij}\phi_{,i}^2 + \varsigma\phi^2\right)dV$ occurring in the expression (43) is always positive. Thus the expression (43) vanishes for t=0, due to the homogeneous initial conditions, and it must be always non-positive for t>0.

Using inequality (37), it follows immediately that the expression (43) must be identically zero for t>0. We thus have

$$u_i = T = \phi = e_{ij} = \sigma_{ij} = 0.$$

This proves the uniqueness of the solution to the complete system of field equations subjected to the initial and boundary conditions.

5. Reciprocity theorem

We shall consider a homogeneous anisotropic fractional order generalized thermoelastic body with voids occupying the region V and bounded by the surface A. We assume that the stresses σ_{ij} and the strains e_{ij} are continuous together with their first derivatives whereas the displacements u_i, temperature T and the volume fraction field ϕ are continuous and have continuous derivatives up to the second order, for $x \in V + A, t > 0$.

The components of surface traction, normal component of the heat flux and the normal component of equilibrated stress vector at regular points of ∂V, are given by

$$T_i = \sigma_{ij}n_j, q = K_{ij}T_{,j}n_i, h = h_i n_i, \tag{44}$$

respectively. We denote by n_j the outward unit normal of ∂V.

To the system of field equation, we must adjoin boundary conditions and initial conditions. We consider the following boundary conditions:

$$u_i(x,t) = U_i(x,t), T(x,t) = \nu(x,t), \phi(x,t) = \varphi(x,t), \tag{45}$$

for all $x \in A, t > 0$;
and the homogeneous initial conditions

$$u_i(x,0) = \dot{u}_i(x,0) = 0, T(x,0) = \dot{T}(x,0) = 0, \phi(x,0) = \dot{\phi}(x,0) = 0, \tag{46}$$

for all $x \in V, t = 0$.

We derive the dynamic reciprocity relationship for a fractional order generalized thermoelastic body, which satisfies Eqs. (6), (7), (12)-(17), the boundary conditions (45) and the homogeneous initial conditions (46), and subjected to the action of body forces $F_i(x,t)$ surface traction $h_i(x,t)$, the volume fraction field $\phi(x,t)$ and the heat flux $q(x,t)$.

Performing the Laplace integral transform defined as

$$\overline{f}(x,s) = L(f(x,t)) = \int_0^\infty f(x,t)e^{-st}dt \tag{47}$$

on Eqs. (6), (7), (12)-(17) and omitting the bars for simplicity, we obtain

$$\sigma_{ij} = c_{ijlm}e_{lm} + B_{ij}\phi - \beta_{ij}T, \tag{48}$$

$$h_i = A_{ij}\phi_{,j}, \tag{49}$$

$$g = -B_{ij}e_{,ij} - \varsigma\phi + mT, \tag{50}$$

$$\rho\eta = \beta_{ij}e_{,ij} + m\phi + aT, \tag{51}$$

$$\sigma_{ij,j} + \rho F_i = \rho s^2 u_i, \tag{52}$$

$$q_i\left[1 + \frac{\tau_0^\alpha}{\alpha!}s^\alpha\right] = -K_{ij}T_{,j}, \tag{53}$$

$$\rho C^*\left[s+\frac{\tau_0^\alpha}{\alpha!}s^{\alpha+1}\right]T+T_0\left[s+\frac{\tau_0^\alpha}{\alpha!}s^{\alpha+1}\right]\left(\beta_{ij}u_{i,j}+m\phi\right)=K_{ij}T_{,ij}, \tag{54}$$

$$h_{i,i}+g+\rho l=\rho\chi s^2\phi. \tag{55}$$

We now consider two problems where applied body forces, volume fraction field and the surface temperature are specified differently. Let the variables involved in these two problems be distinguished by superscripts in parentheses. Thus, we have $u_i^{(1)},e_{ij}^{(1)},\sigma_{ij}^{(1)},T^{(1)},\phi^{(1)},....$ for the first problem and $u_i^{(2)},e_{ij}^{(2)},\sigma_{ij}^{(2)},T^{(2)},\phi^{(2)},....$ for the second problem. Each set of variables satisfies the system of equations (48)-(55).

Using the strain-displacement relation, the assumption $\sigma_{ij}=\sigma_{ji}$ and the divergence theorem, with the aid of (44) and (52), we obtain

$$\int_V \sigma_{ij}^{(1)}e_{ij}^{(2)}dV=\int_A T_i^{(1)}u_i^{(2)}dA-\rho\int_V s^2u_i^{(1)}u_i^{(2)}dV+\rho\int_V F_i^{(1)}u_i^{(2)}dV. \tag{56}$$

A similar expression is obtained for the integral $\int_V \sigma_{ij}^{(2)}e_{ij}^{(1)}dV$, from which together with Eq. (56), it follows that

$$\int_V\left[\sigma_{ij}^{(1)}e_{ij}^{(2)}-\sigma_{ij}^{(2)}e_{ij}^{(1)}\right]dV=\int_A\left[T_i^{(1)}u_i^{(2)}-T_i^{(2)}u_i^{(1)}\right]dA+\rho\int_V\left[F_i^{(1)}u_i^{(2)}-F_i^{(2)}u_i^{(1)}\right]dV. \tag{57}$$

Now multiplying equation (48) by $e_{ij}^{(2)}$ and $e_{ij}^{(1)}$ for the first and second problem respectively, Subtracting and integrating over the region V and using the symmetry properties of c_{ijlm}, we obtain

$$\int_V\left[\sigma_{ij}^{(1)}e_{ij}^{(2)}-\sigma_{ij}^{(2)}e_{ij}^{(1)}\right]dV=\int_V B_{ij}\left[\phi^{(1)}e_{ij}^{(2)}-\phi^{(2)}e_{ij}^{(1)}\right]dV-\int_V\beta_{ij}\left[T^{(1)}e_{ij}^{(2)}-T^{(2)}e_{ij}^{(1)}\right]dV. \tag{58}$$

Equating Eqs. (57) and (58), we get the first part of the reciprocity theorem

$$\int_A\left[T_i^{(1)}u_i^{(2)}-T_i^{(2)}u_i^{(1)}\right]dA+\rho\int_V\left[F_i^{(1)}u_i^{(2)}-F_i^{(2)}u_i^{(1)}\right]dV=$$

$$=\int_V B_{ij}\left[\phi^{(1)}e_{ij}^{(2)}-\phi^{(2)}e_{ij}^{(1)}\right]dV-\int_V\beta_{ij}\left[T^{(1)}e_{ij}^{(2)}-T^{(2)}e_{ij}^{(1)}\right]dV, \tag{59}$$

which contains the mechanical causes of motion F_i,T_i.

From equation (54), we obtain

$$\rho C^*\left[s+\frac{\tau_0^\alpha}{\alpha!}s^{\alpha+1}\right]T+T_0\left[s+\frac{\tau_0^\alpha}{\alpha!}s^{\alpha+1}\right]\left(\beta_{ij}u_{i,j}+m\phi\right)=K_{ij}T_{,ij}, \tag{60}$$

Now multiplying equation (60) by $T^{(2)}$ and $T^{(1)}$ for the first and second problem respectively, subtracting and integrating over the region V, we obtain

$$\int_V\left[K_{ij}\left(T_{,j}^{(1)}T^{(2)}-T_{,j}^{(2)}T^{(1)}\right)\right]_{,i}dV=$$

$$= T_0\left[s+\frac{\tau_0^{\alpha}}{\alpha!}s^{\alpha+1}\right]\int_V \beta_{ij}\left(e_{,ij}^{(1)}T^{(2)}-e_{,ij}^{(2)}T^{(1)}\right)dV + T_0\left[s+\frac{\tau_0^{\alpha}}{\alpha!}s^{\alpha+1}\right]\int_V m\left(\phi^{(1)}T^{(2)}-\phi^{(2)}T^{(1)}\right)dV\,. \quad (61)$$

Using divergence theorem and with the aid of (44) and (45), equation (61) becomes

$$\int_V \left(q^{(1)}\nu^{(2)}-q^{(2)}\nu^{(1)}\right)dV =$$

$$= T_0\left[s+\frac{\tau_0^{\alpha}}{\alpha!}s^{\alpha+1}\right]\int_V \beta_{ij}\left(e_{ij}^{(1)}T^{(2)}-e_{ij}^{(2)}T^{(1)}\right)dV + T_0\left[s+\frac{\tau_0^{\alpha}}{\alpha!}s^{\alpha+1}\right]\int_V m\left(\phi^{(1)}T^{(2)}-\phi^{(2)}T^{(1)}\right)dV \quad (62)$$

Equation (62) constitutes the second part of the reciprocity theorem which contains the thermal causes of motion ν and q.

Using equation (49) and (50) in (55), we obtain

$$A_{ij}\phi_{,ii} - B_{ij}e_{,ij} - \xi\phi + mT + \rho l = \rho\chi s^2\phi. \quad (63)$$

Now multiplying equation (63) by $\phi^{(2)}$ and $\phi^{(1)}$ for the first and second problem respectively, subtracting and integrating over the region V, we obtain

$$\int_V A_{ij}\left(\phi_{,ii}^{(1)}\phi^{(2)}-\phi_{,ii}^{(2)}\phi^{(1)}\right)dV =$$

$$= \int_V B_{ij}\left(e_{,ij}^{(1)}\phi^{(2)}-e_{,ij}^{(2)}\phi^{(1)}\right)dV - m\int_V\left(T^{(1)}\phi^{(2)}-T^{(2)}\phi^{(1)}\right)dV - \int_V \rho l\left(\phi^{(2)}-\phi^{(1)}\right). \quad (64)$$

Using divergence theorem and with the aid of (15), (44) and (45), Eq. (64) becomes

$$\int_A \left(h^{(1)}\phi^{(2)}-h^{(2)}\phi^{(1)}\right)dA =$$

$$= \int_V B_{ij}\left(e_{ij}^{(1)}\phi^{(2)}-e_{ij}^{(2)}\phi^{(1)}\right)dV - m\int_V\left(T^{(1)}\phi^{(2)}-T^{(2)}\phi^{(1)}\right)dV - \int_V \rho l\left(\phi^{(2)}-\phi^{(1)}\right)dV, \quad (65)$$

The equation (65) constitutes the third part of reciprocity theorem which contains volume fraction field ϕ.

Eliminating the integrals $\int_V B_{ij}\left(e_{,ij}^{(1)}\phi^{(2)}-e_{,ij}^{(2)}\phi^{(1)}\right)dV$ and $\int_V \beta_{ij}\left(e_{,ij}^{(1)}T^{(2)}-e_{,ij}^{(2)}T^{(1)}\right)dV$ from equations (59), (62) and (65), we obtain

$$T_0\left(s+\frac{\tau_0^{\alpha}}{\alpha!}s^{\alpha+1}\right)\int_A\left[T_i^{(1)}u_i^{(2)}-T_i^{(2)}u_i^{(1)}\right]dA + \rho T_0\left(s+\frac{\tau_0^{\alpha}}{\alpha!}s^{\alpha+1}\right)\int_V\left[F_i^{(1)}u_i^{(2)}-F_i^{(2)}u_i^{(1)}\right]dV -$$

$$-T_0\left(s+\frac{\tau_0^{\alpha}}{\alpha!}s^{\alpha+1}\right)\int_A\left(h^{(1)}\varphi^{(2)}-h^{(2)}\varphi^{(1)}\right)dA - T_0\left(s+\frac{\tau_0^{\alpha}}{\alpha!}s^{\alpha+1}\right)\int_V \rho l\left(\phi^{(2)}-\phi^{(1)}\right)dV - \int_V\left(q^{(1)}\nu^{(2)}-q^{(2)}\nu^{(1)}\right)dV = 0, \quad (66)$$

This is the general reciprocity theorem in the Laplace transform domain.

To invert the Laplace transform in the eqs.(59), (62), (65) and (66) we shall use the convolution theorem

$$L^{-1}\{F(s)G(s)\} = \int_0^t f(t-\xi)g(\xi)d\xi = \int_0^t g(t-\xi)f(\xi)d\xi \tag{67}$$

and the symbolic notations

$$M(f) = 1 + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha f(x,\xi)}{\partial \xi^\alpha}. \tag{68}$$

Inverting Eq. (59), we obtain the first part of the reciprocity theorem in the final form

$$\int_A\int_0^t T_i^{(1)}(x,t-\xi)u_i^{(2)}(x,\xi)d\xi dA + \rho\int_V\int_0^t F_i^{(1)}(x,t-\xi)u_i^{(2)}(x,\xi)d\xi dV -$$

$$-\int_V\int_0^t B_{ij}\phi^{(1)}(x,t-\xi)e_{ij}^{(2)}(x,\xi)d\xi dV + \int_V\int_0^t \beta_{ij}T^{(1)}(x,t-\xi)e_{ij}^{(2)}(x,\xi)d\xi dV = S_{21}^{12}. \tag{69}$$

Here S_{21}^{12} indicates the same expression as on the left-hand side except that the superscripts (1) and (2) are interchanged.

Inverting Eq. (62), we obtain the second part of reciprocity theorem in the final form

$$\int_A\int_0^t q^{(1)}(x,t-\xi)\nu^{(2)}(x,\xi)d\xi dA + T_0\int_V\int_0^t \beta_{ij}T^{(1)}(x,t-\xi)\frac{\partial M\left(e_{ij}^{(2)}\right)}{\partial \xi}(x,\xi)d\xi dV +$$

$$+T_0\int_V\int_0^t mT^{(1)}(x,t-\xi)\frac{\partial M\left(\phi^{(2)}\right)}{\partial \xi}(x,\xi)d\xi dV = S_{21}^{12}. \tag{70}$$

Inverting Eq.(65), we obtain the third part of reciprocity theorem in the final form

$$\int_A\int_0^t h^{(1)}(x,t-\xi)\varphi^{(2)}(x,\xi)d\xi dA - \int_V\int_0^t B_{ij}e_{ij}^{(1)}(x,t-\xi)\phi^{(2)}(x,\xi)d\xi dV +$$

$$+m\int_V\int_0^t T^{(1)}(x,t-\xi)\phi^{(2)}(x,\xi)d\xi dV + \rho l\int_V\int_0^t \phi^{(2)}(x,\xi)dV = S_{21}^{12}, \tag{71}$$

Finally, inverting Eq.(66), we obtain the general reciprocity theorem in the final form

$$T_0\int_A\int_0^t T_i^{(1)}(x,t-\xi)\frac{\partial M\left(u_i^{(2)}\right)}{\partial \xi}(x,\xi)d\xi dA + \rho T_0\int_V\int_0^t F_i^{(1)}(x,t-\xi)\frac{\partial M\left(u_i^{(2)}\right)}{\partial \xi}(x,\xi)d\xi dV +$$

$$+T_0\int_A\int_0^t h^{(1)}(x,t-\xi)\frac{\partial M\left(\varphi^{(2)}\right)}{\partial \xi}(x,\xi)d\xi dA + T_0\rho l\int_V\left(\frac{\partial}{\partial t} + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)\phi^{(2)}(x,\xi)dV -$$

$$-\int_A\int_0^t q^{(1)}(x,t-\xi)\nu^{(2)}(x,\xi)d\xi dA = S_{21}^{12}. \tag{72}$$

6. Orthotropic media

The equations (6),(7) and (13) with the aid of (14)-(16) without extrinsic equilibrated force, body force and heat sources for fractional order generalized thermoelastic orthotropic medium with voids are,

$$c_{11}u_{1,11} + (c_{66} + c_{12})u_{2,12} + c_{66}u_{1,22} + (c_{13} + c_{55})u_{3,13} + c_{55}u_{1,33} + B_1\phi_{,1} - \beta_1 T_{,1} = \rho\ddot{u}_1, \tag{73}$$

$$c_{22}u_{2,22} + c_{66}u_{2,11} + (c_{66} + c_{21})u_{1,21} + (c_{23} + c_{44})u_{3,32} + c_{44}u_{2,33} + B_2\phi_{,2} - \beta_2 T_{,2} = \rho\ddot{u}_2, \tag{74}$$

$$c_{55}u_{3,31} + c_{44}u_{3,22} + c_{33}u_{3,33} + (c_{55} + c_{31})u_{1,13} + (c_{32} + c_{44})u_{2,23} + B_3\phi_{,3} - \beta_3 T_{,3} = \rho\ddot{u}_3, \tag{75}$$

$$-\xi\phi + mT + (\phi_{,11}A_1 + \phi_{,22}A_2 + \phi_{,33}A_3) - (u_{1,1}B_1 + u_{2,2}B_2 + u_{3,3}B_3) = \rho\chi\ddot{\phi}, \tag{76}$$

$$K_1T_{,11} + K_2T_{,22} + K_3T_{,33} =$$

$$\rho C^*\left(1 + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)\dot{T} + T_0\left(1 + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)(\dot{u}_{1,1}\beta_1 + \dot{u}_{2,2}\beta_2 + \dot{u}_{3,3}\beta_3 + m\dot{\phi}), \tag{77}$$

where $A_{ij} = A_i\delta_{ij}$, $B_{ij} = B_i\delta_{ij}$, $\beta_{ij} = \beta_i\delta_{ij}$, $K_{ij} = K_i\delta_{ij}$, i is not summed.

In the above equations (73)-(77), we use the contracting subscript notations $11\to1$, $22\to2$, $33\to3$, $23\to4$, $13\to5$, $12\to6$ to relate c_{ijkm} to c_{ln} (i,j,k,m=1,2,3 and l,n=1,2,3,4,5,6).
Now we will discuss two-dimensional plane wave propagation in homogeneous, orthotropic generalized thermoelastic medium with voids. For two dimensional problem, we have

$$u = (u_1, 0, u_3), \quad \phi(x_1, 0, x_3), \quad T(x_1, 0, x_3). \tag{78}$$

We define the following dimensionless quantities

$$x_i' = \frac{\omega_1 x_i}{c_1}, \quad u_i' = \frac{\omega_1 u_i}{c_1}, \quad t' = \omega_1 t, \quad \tau_0' = \omega_1\tau_0, \quad \phi' = \frac{\omega_1^2\chi\phi}{c_1^2}, \quad T' = \frac{\beta_1 T}{c_{11}}, \tag{79}$$

where $c_1^2 = \dfrac{c_{11}}{\rho}$, $\omega_1 = \dfrac{C^* c_{11}}{K_1}$.

Upon introducing the dimensionless quantities defined by equation (79) in equations (73)-(77) and with the aid of (78) and after suppressing the primes, we obtain

$$u_{1,11} + \delta_1 u_{1,33} + \delta_2 u_{3,13} + \delta_3\phi_{,1} - T_{,1} = \ddot{u}_1, \tag{80}$$

$$\delta_2^* u_{2,33} + \delta_1^* u_{2,11} = \ddot{u}_2, \tag{81}$$

$$\delta_1 u_{3,11} + \delta_4 u_{3,33} + \delta_2 u_{1,13} + \delta_5\phi_{,3} - \bar{\beta}T_{,3} = \ddot{u}_3, \tag{82}$$

$$-\delta_7\phi + \phi_{,11}\delta_8 + \phi_{,33}\delta_9 - u_{1,1}\delta_{10} - u_{3,3}\delta_{11} + T\delta_{12} = \ddot{\phi}, \tag{83}$$

$$\delta_{13}\tau^{12}\dot{T} + \delta_{14}\tau^{12}(\dot{u}_{1,1} + \bar{\beta}\dot{u}_{3,3}) + \delta_{15}\tau^{12}\dot{\phi} = T_{,11} + \bar{K}T_{,33}, \tag{84}$$

where $\delta_1 - \delta_5, \delta_7 - \delta_{15}, \tau^{12}$ are given in Appendix.

The equation (81) corresponds to purely quasi-transverse wave mode that decouples from the rest of the motion and is not affected by the voids and thermal.

7. Solution of the problem

For plane harmonic waves, we assume the solution of equations (80)-(84) of the form

$$(u_1, u_2, u_3, T, \phi) = (U_1, U_2, U_3, T^*, \phi^*)\exp[\iota(\xi(x_1 l_1 + x_3 l_3) - \omega t)], \tag{85}$$

where ω is the circular frequency and ξ is the complex wave number. U_1, U_2, U_3, T^* and ϕ^* are undetermined amplitude vectors that are independent of time t and coordinates x_i (i=1, 3), l_1 and l_3 are the direction cosines of the wave normal onto the x_1-x_3 plane with the property $l_1^2 + l_3^2 = 1$.

Upon using solutions (85) in the equations (80),(82)-(84), we obtain

$$\left[-\xi^2 l_1^2 - \xi^2 l_3^2 \delta_1 + \omega^2\right]U_1 + \left[-\xi^2 l_1 l_3 \delta_2\right]U_3 + \iota\xi l_1 l_3 \delta_3 \phi^* - \iota\xi l_1 T^* = 0, \tag{86}$$

$$\left[-\xi^2 l_1 l_3 \delta_2\right]U_1 + \left[-\xi^2 l_1^2 \delta_1 - \xi^2 l_3^2 \delta_4 + \omega^2\right]U_3 + \iota\xi l_3 \delta_5 \phi^* - \iota\xi l_3 \overline{\beta} T^* = 0, \tag{87}$$

$$\left[-\iota\xi l_1 \delta_{10}\right]U_1 + \left[-\iota\xi l_3 \delta_{11}\right]U_3 + \left[-\xi^2 l_1^2 \delta_8 - \xi^2 l_3^2 \delta_9 + \omega^2 - \delta_7\right]\phi^* + \delta_{12} T^* = 0, \tag{88}$$

$$\left[\xi l_1 \omega \delta_{14} \tau^{13}\right]U_1 + \left[\xi l_3 \delta_{14} \omega \tau^{13} \overline{\beta}\right]U_3 - \iota\omega\delta_{15}\tau^{13}\phi^* + \left[\xi^2 l_1^2 + \xi^2 l_3^2 \overline{K} - \iota\omega\delta_{13}\tau^{13}\right]T^* = 0, \tag{89}$$

where $\tau^{13} = 1 + \dfrac{\tau_0^\alpha}{\alpha!}(-\iota\omega)^\alpha$ are given in Appendix.

The non-trivial solution of the system of equations (86)-(89) is ensured by a determinantal equation given by

$$\begin{vmatrix} -\xi^2 l_1^2 - \xi^2 l_3^2 \delta_1 + \omega^2 & -\xi^2 l_1 l_3 \delta_2 & \iota\xi l_1 l_3 & -\iota\xi l_1 \\ -\xi^2 l_1 l_3 \delta_2 & -\xi^2 l_1^2 \delta_1 - \xi^2 l_3^2 \delta_4 + \omega^2 & \iota\xi l_3 \delta_5 & -\iota\xi l_3 \overline{\beta} \\ -\iota\xi l_1 \delta_{10} & -\iota\xi l_3 \delta_{11} & -\xi^2 l_1^2 \delta_8 - \xi^2 l_3^2 \delta_9 + \omega^2 - \delta_7 & \delta_{12} \\ \xi l_1 \omega \delta_{14} \tau^{13} & \xi l_3 \delta_{14} \omega \tau^{13} \overline{\beta} & -\iota\omega\delta_{15}\tau^{13} & \xi^2 l_1^2 + \xi^2 l_3^2 \overline{K} - \iota\omega\delta_{13}\tau^{13} \end{vmatrix} = 0. \tag{90}$$

The equation (90) yields to following polynomial characteristic equation in ξ as

$$A\xi^8 + B\xi^6 + C\xi^4 + D\xi^2 + E = 0, \tag{91}$$

where the coefficients A, B, C, D, E and $\overline{\beta}, \overline{K}$ are given in Appendix. Solving equation (91), we obtain eight roots of ξ, that is, $\pm\xi_1, \pm\xi_2, \pm\xi_3$ and $\pm\xi_4$. Corresponding to these roots, there exist four waves in descending order of their velocities, namely a quasi-longitudinal (qP) wave, quasi-transverse qS wave, quasi-longitudinal thermal (qT) wave and a quasi-longitudinal volume fractional (qV) wave.

Now we derive the expressions of phase velocity, attenuation coefficient, specific loss and penetration depth of these types of waves as:

Phase velocity.

The phase velocity is given by

$$V_i = \frac{\omega}{\left|\operatorname{Re}(\xi_i)\right|}, i = 1,2,3,4, \tag{92}$$

where V_i $(i = 1,2,3,4)$ are, respectively, the velocities of qP, qS, qT and qV waves.

Attenuation coefficient.

The attenuation coefficient is defined as

$$Q_i = \operatorname{Im}(\xi_i), \; i = 1,2,3,4, \tag{93}$$

where Q_i $(i = 1,2,3,4)$ are, respectively, the attenuation coefficients of qP, qS, qT and qV waves.

Specific loss.

The specific loss is the ratio of energy (ΔW) dissipated in taking a specimen through a stress cycle, to the elastic energy (W) stored in the specimen when the strain is a maximum. The specific loss is the most direct method of defining internal friction for a material. For a sinusoidal plane wave of small amplitude, Kolksy [34] shows that the specific loss $\Delta W / W$ equals 4π times the absolute value of the imaginary part of ξ to that of real part of ξ i.e.

$$R_i = \left(\frac{\Delta W}{W}\right)_i = 4\pi\left|\frac{\operatorname{Im}(\xi_i)}{\operatorname{Re}(\xi_i)}\right|, \; i = 1,2,3,4. \tag{94}$$

Penetration depth.

The Penetration depth is defined by

$$S_i = \frac{1}{\operatorname{Im}(\xi_i)}, \; i = 1,2,3,4. \tag{95}$$

Particular cases.

(1) In the absence of void effect i.e. when $m = A_1 = A_3 = B_1 = B_3 = 0, \; \varsigma = 0,$ the characteristic equation (91) reduces to the characteristic equation corresponding to the orthotropic fractional order generalized thermoelastic medium:

$$A^*\xi^6 + B^*\xi^4 + C^*\xi^2 + D^* = 0,$$

where A^*, B^*, C^* and D^* are given in Appendix.

(2) If α=1 in equation (12),(13) and (89), the corresponding results reduce to the case of Lord-Shulman theory of generalized thermoelasticity with voids.

8. Numerical results and discussion

In this section, numerical discussion for phase velocities, attenuation coefficient, specific loss and penetration depth of quasi-longitudinal (qP) wave , quasi-transverse waves (qS), quasi-longitudinal thermal (qT) wave, quasi-longitudinal volume fractional (qV) wave and a is presented.

The material chosen for this purpose is Cobalt, whose physical data are given by [35]:

$c_{11} = 3.071 \times 10^{11}\ N/m^2, \quad c_{12} = 1.650 \times 10^{11}\ N/m^2, \quad c_{13} = 1.027 \times 10^{11}\ N/m^2,$

$c_{33} = 3.581 \times 10^{11}\ N/m^2, \; c_{44} = 1.791 \times 10^{11}\ N/m^2, \; T_0 = 298K, \; \tau_0 = 0.04\ s, \rho = 8.836 \times 10^3\ Kg/m^3,$

β_1=7.04 × 10^5 N/m^2deg, β_3= 6.90 × 10^6 N/m^2deg, C^*=4.27 × 10^2 J/Kg deg ,
K_1=0.690 × 10^2 W/m deg, K_3=0.690 × 10^2 W/m deg, ω_1=1.88 × 10^{12}.

Void parameters are $\chi = 3.05655 \times 10^{-15} m^2$, $A_1 = 14.798 \times 10^{-5}$ N, $A_3 = 13.9174 \times 10^{-5}$ N, $\varsigma = 3.41 \times 10^{10}$ N / m^2, B_1=8.52849 × 10^{10} N, B_3=7.41 × 10^{10} N, m=1.23849 × 10^5 N / m^2K^{-1}.

We can solve equation (91) with the help of the software Matlab 7.0.4 and after solving the equation (91) and using the formulas given by (92)-(95), we can compute the values of phase velocity, attenuation coefficient, specific loss and penetration depth for intermediate values of frequency (ω) and different values of fractional order derivative i.e. α = 0.5, 1.0, 1.5. In all the figures horizontal lines, square boxes and vertical lines corresponds to α = 0.5, 1.0, and 1.5 respectively.

Phase velocity.

Figure 1 shows that for all fractional orders, the values of V_1 increase smoothly with increase in values of ω. On comparing the values of V_1 for different fractional orders, the values of V_1 increase with increase in fractional order. It is evident from Figure 2 that firstly the values of V_2 decrease smoothly but lastly remain constant.

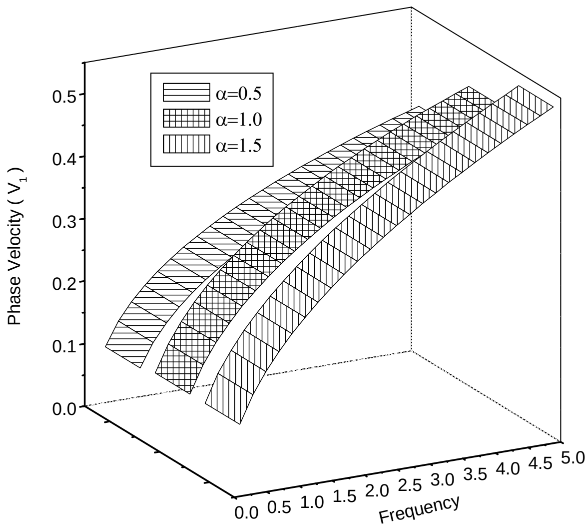

Fig. 1. Variation of phase velocity V_1 w.r.t. frequency.

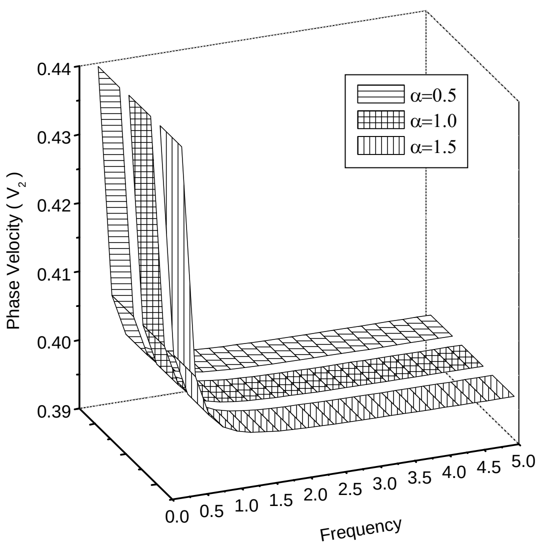

Fig. 2. Variation of phase velocity V_2 w.r.t. frequency.

Figure 3 shows that the values of V_3 first increase rapidly and then decrease smoothly and shows the constant behavior. Figure 4 indicates that values of V_4 first decrease rapidly and finally constant behavior is noticed.

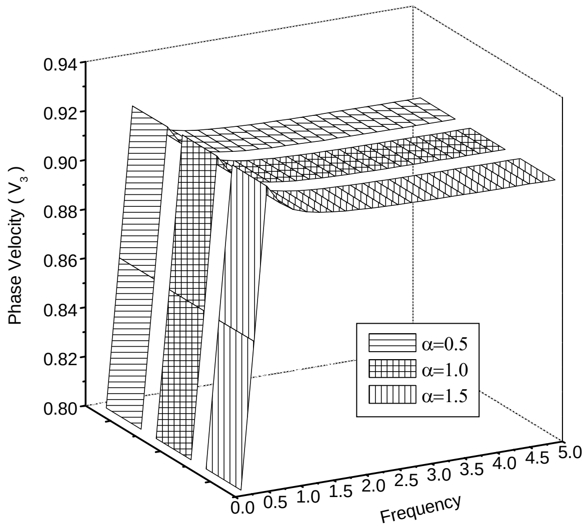

Fig. 3. Variation of phase velocity V_3 w.r.t. frequency.

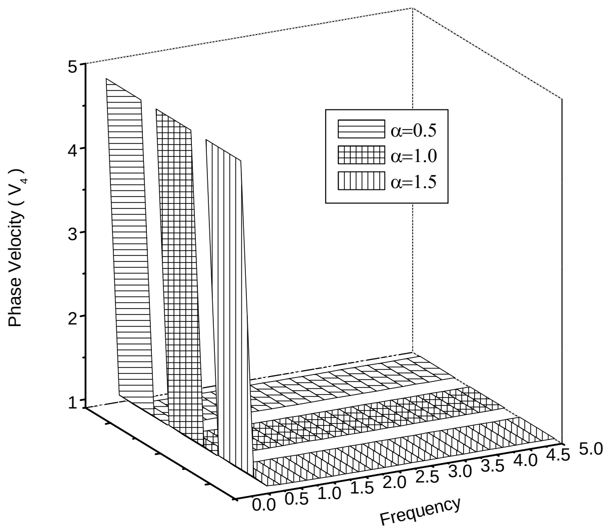

Fig. 4. Variation of phase velocity V_4 w.r.t. frequency.

Attenuation Coefficient.

It is noticed from Fig. 5 that values of Q_1 increase smoothly with increase in values of ω. The values of Q_1 increase with increase in fractional order α. Figure 6 shows that Q_2 first decrease and then increase smoothly with increase in ω and finally becomes constant. From Fig. 7, it is evident that Q_3 increase with increase in ω. Figure 8 indicates that Q_4 shows the behavior opposite to that of Q_3.

Specific loss.

Figure 9 shows that for α=0.5, R_1 decrease smoothly for $\omega < 3$ and becomes constant for $\omega \geq 3$ whereas for α=1.0, it decrease rapidly and for α=1.5, it decrease slowly. The values of R_1 increase with increase in fractional order α. Figure 10 indicates that R_2 first decrease rapidly and lastly becomes constant. The values of R_1 decrease with increase in fractional order α and maximum value occurs for α=0.5. It is evident from Fig. 11 that the values of R_3 increase with increase in ω and lastly shows the constant behavior. Figure 12 shows that the behavior of the values of R_4 is same as that of R_3.

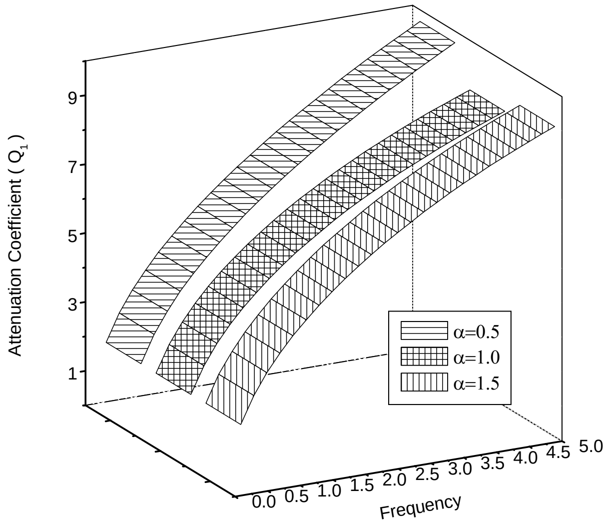

Fig. 5. Variation of attenuation coefficient Q_1 w.r.t. frequency.

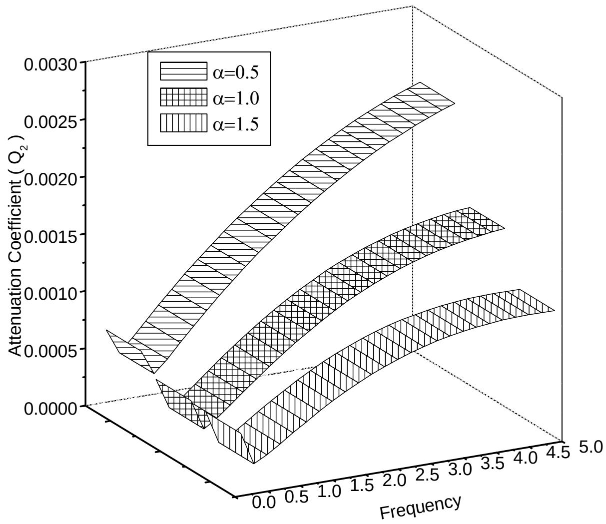

Fig. 6. Variation of attenuation coefficient Q_2 w.r.t. frequency.

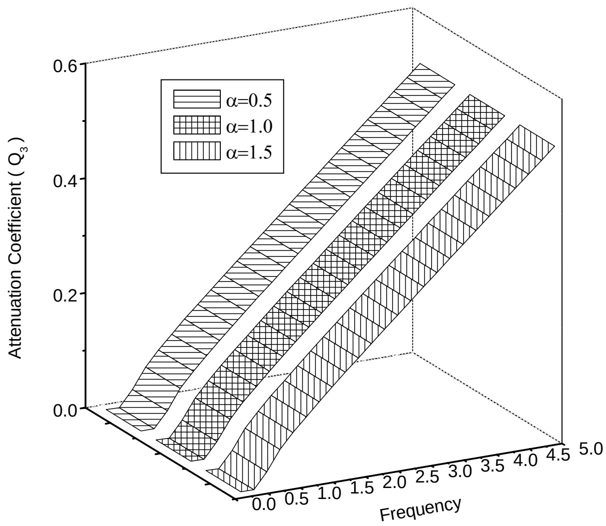

Fig. 7. Variation of attenuation coefficient Q_3 w.r.t. frequency.

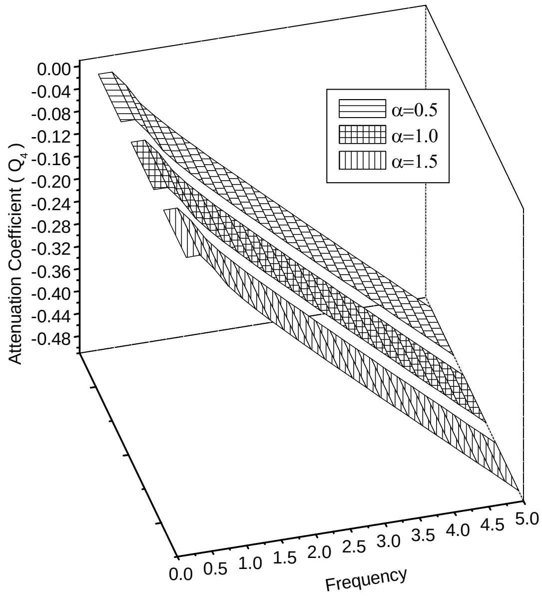

Fig. 8. Variation of attenuation coefficient Q_4 w.r.t. frequency.

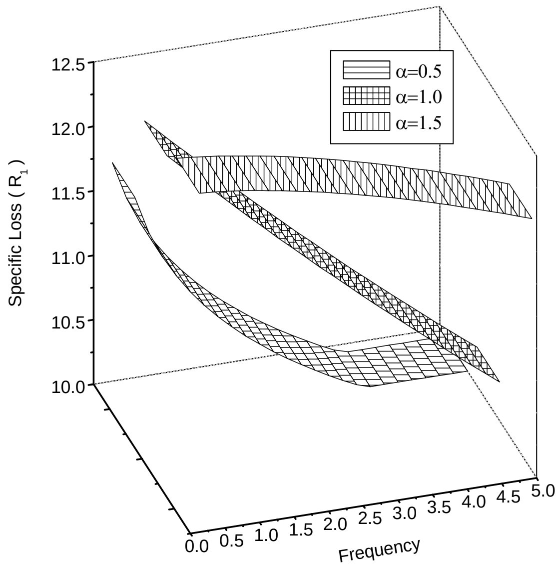

Fig. 9. Variation of specific loss R_1 w.r.t. frequency.

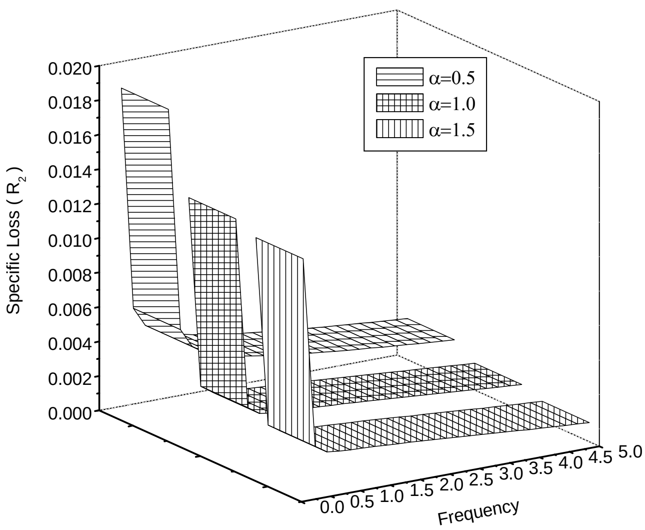

Fig. 10. Variation of specific loss R_2 w.r.t. frequency.

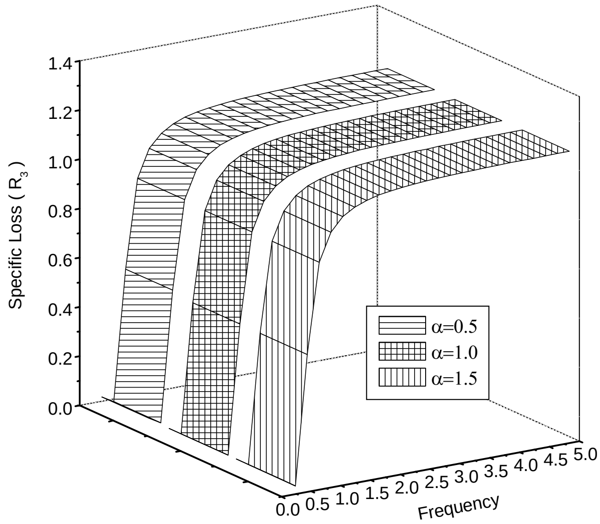

Fig. 11. Variation of specific loss R_3 w.r.t. frequency.

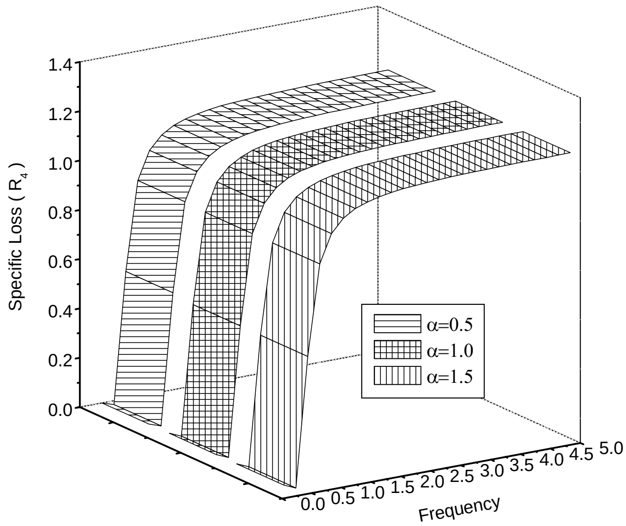

Fig. 12. Variation of specific loss R_4 w.r.t. frequency.

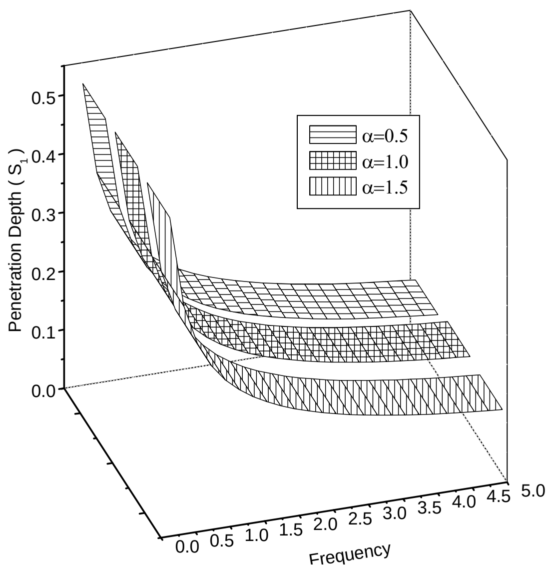

Fig. 13. Variation of penetration depth S_1 w.r.t. frequency.

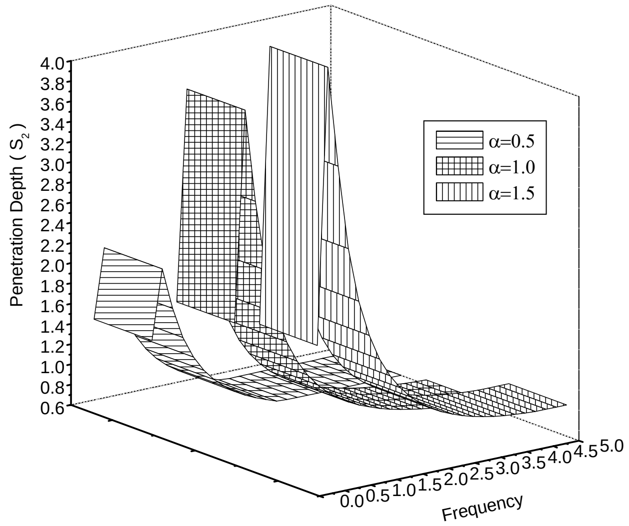

Fig. 14. Variation of penetration depth S_2 w.r.t. frequency.

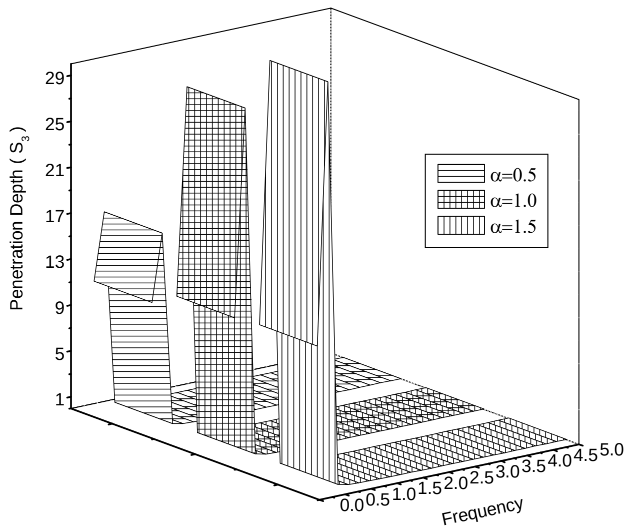

Fig. 15. Variation of penetration depth S_3 w.r.t. frequency.

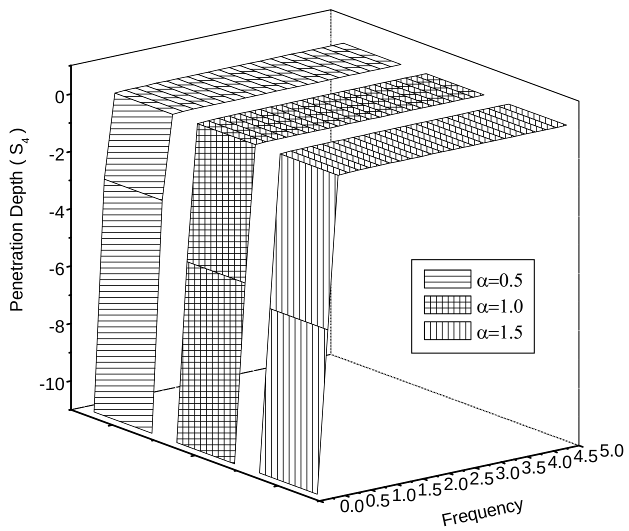

Fig. 16. Variation of penetration depth S_4 w.r.t. frequency.

Penetration depth.

Figure 13 shows that penetration depth S_1 decrease smoothly with increase in values of ω. Figure 14 indicates that S_2 increase rapidly for small values of ω and finally constant behavior is noticed. The values of S_1 increase with increase in fractional order α. It is evident from Fig. 15, that S_3 rapidly increase and decrease initially and finally becomes constant. The values of S_3 increase with increase in fractional order α. Figure 16 depicts that the behavior of the values of S_4 is opposite to that of V_4 in Fig. 4. The values of penetration depth are magnified by multiplying S_2, S_3, and S_4 by 10^3, 10^2, and 10^3 respectively.

9. Conclusions

A model of anisotropic, homogeneous thermoelastic solids with fractional order derivative and voids based on the theory of Lord and Shulman is given. Using the variational theorem, the uniqueness theorem of solution of the initial boundary value problem is proved, and the dynamic reciprocity theorem is derived for the given model.

The governing equations for the orthotropic thermoelastic material with fractional order derivative and voids are presented. For two dimensional problem there exist quasi-longitudinal wave (qP) , quasi-transverse waves (qS), quasi-longitudinal thermal wave (qT) and a quasi-longitudinal volume fractional wave (qV). The phase velocity, attenuation coefficient, specific loss and penetration depth are computed numerically and presented graphically. Some particular cases are also discussed.

From Figures it is observed that values of phase velocity V_1 increase, attenuation coefficients Q_1 and Q_2 decrease, specific loss R_1 increase, specific loss R_2 decrease, penetration depth S_2 and S_3 increase with increase in the fractional order α.

Appendix

$$\delta_1 = \frac{c_{55}}{c_{11}},\quad \delta_1^* = \frac{c_{66}}{c_{11}},\quad \delta_2 = \frac{(c_{55}+c_{13})}{c_{11}},\quad \delta_2^* = \frac{c_{44}}{c_{11}},\quad \delta_3 = \frac{B_1}{\chi\rho\omega_1^2},\quad \delta_4 = \frac{c_{33}}{c_{11}},\quad \delta_5 = \frac{B_3}{\chi\rho\omega_1^2},$$

$$\delta_7 = \frac{\varsigma}{\chi\rho\omega_1^2},\quad \delta_8 = \frac{A_1}{\chi c_{11}},\quad \delta_9 = \frac{A_3}{\chi c_{11}},\quad \delta_{10} = \frac{B_1}{c_{11}},\quad \delta_{11} = \frac{B_3}{c_{11}},\quad \delta_{12} = \frac{m}{\beta_1},\quad \delta_{13} = \frac{\rho C^* c_1^2}{K_1\omega_1},$$

$$\delta_{14} = \frac{T_0\beta_1^2}{\rho K_1\omega_1},\quad \delta_{15} = \frac{T_0\beta_1 m c_1^2}{\rho K_1\omega_1^3\chi},\quad \tau^{12} = 1 + \frac{\tau_0^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha},\quad \bar{\beta} = \frac{\beta_3}{\beta_1},\quad \bar{K} = \frac{K_3}{K_1},$$

$A = R_2 R_7 R_{13} R_{19} + R_3 R_6 R_{13} R_{19}$,

$B = R_2 M_1 + R_1 R_7 R_{13} R_{19} + [R_6 \{R_3 (R_{13} R_{18} + R_{12} R_{19}) - R_4 R_{11} R_{19} - R_5 R_{13} R_{16}\}] +$

$+R_{10} R_{19} (R_3 R_8 - R_4 R_7) + R_{13} R_{15} (R_5 R_7 - R_3 R_9)$,

$C = R_1M_1 + R_2M_2 [R_6 \{R_3R_{12}R_{18} - R_{11} (R_4R_{18} - R_5R_{17})\}] + [R_{10} \{R_3 (R_8R_{18} - R_9R_{17}) -$

$R_7 (R_4R_{18} - R_5R_{17}) - R_1R_4R_{19} + R_{16} (R_4R_9 - R_5R_8)\}] + [R_{15} \{R_3 (R_8R_{14} - R_9R_{12}) -$

$-R_7 (R_4R_{14} - R_5R_{12}) + +R_1R_5R_{13}R_{15}\}]$,

$D = R_2M_3 + R_1M_2 + R_6R_{16} (R_4R_{14} - R_5R_{12}) + R_1R_{10} (R_5R_{17} - R_4R_{18})$,

$E = R_1M_3$,

$M_1 = R_7(R_{13}R_{18}+R_{12}R_{19}) + R_1R_{13}R_{19} - R_8R_{11}R_{19} - R_9R_{13}R_{16},$

$M_2 = R_7(R_{12}R_{18} - R_{14}R_{17}) + R_1(R_{13}R_{18} + R_{12}R_{19}) + R_{16}(R_8R_{14} - R_9R_{12}) - R_{11}(R_8R_{18}-R_9R_{17}),$

$M_3 = R_1(R_{12}R_{18} - R_{14}R_{17}),$

$A^* = R_{19}\left(R_2R_7 - R_3R_6\right),$

$B^* = R_1R_7R_{19} + R_2\left(R_{18}R_7 + R_1R_{19} - R_9R_{16}\right) - R_6\left(R_3R_{18} - R_5R_{16}\right) + R_{15}\left(R_3R_9 - R_5R_7\right),$

$C^* = R_1\left(R_{18}R_7 + R_1R_{19} - R_9R_{16}\right) + R_1R_2R_{18} - R_1R_5R_{15},\ D^* = R_1^2R_{18},$

$R_1 = \omega^2,\ R_2 = -l_1^2 - l_3^2\delta_1,\ R_3 = -\delta_2 l_1 l_3,\ R_4 = \iota l_1 l_3,\ R_5 = -\iota l_1,\ R_6 = -l_1 l_3\delta_2,$

$R_7 = -l_1^2\delta_1 - l_3^2\delta_4,\ R_8 = \iota l_3\delta_5,\ R_9 = -\iota l_3\overline{\beta},\ R_{10} = -\iota l_1\delta_{10},\ R_{11} = -\iota l_3\delta_{11},$

$R_{12} = \omega^2 - \delta_7,\ R_{13} = -l_1^2\delta_8 - l_3^2\delta_9,\ R_{14} = \delta_{12},\ R_{15} = l_1\omega\delta_{14}\tau^{12},\ R_{16} = l_3\delta_{14}\omega\tau^{12}\overline{\beta},$

$R_{17} = -\iota\omega\delta_{15}\tau^{12},\ R_{18} = -\iota\omega\delta_{13}\tau^{12},\ R_{19} = l_1^2 + l_3^2\overline{K}.$

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