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A STUDY OF PLANE WAVE AND FUNDAMENTAL SOLUTION IN THE THEORY OF AN ELECTRO-MICROSTRETCH GENERALIZED THERMOELASTIC SOLID

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Abstract. The present article deals with the study of propagation of plane wave and fundamental solution in an electro-microstretch generalized thermoelastic solid. It is found that for two dimensional model, there exist three coupled longitudinal waves, namely longitudinal displacement wave (LD-wave), thermal wave (T-wave), longitudinal microstretch wave (LM-wave) and two coupled transverse displacement and transverse microrotational (CD I, CD II) waves. The phase velocity, attenuation coefficient, specific loss and penetration depth are computed numerically and depicted graphically. In addition, we construct the fundamental solution of the system of differential equations in the theory of an electro-microstretch generalized thermoelastic solid in case of steady oscillations in terms of elementary functions. Some basic properties of the fundamental solution are established. Some special cases are also discussed.

1. Introduction

Eringen [1] developed the theory of a micropolar elastic solid with stretch in which he has taken the effect of axial stretch during the rotation of molecules. Eringen [2, 3] also developed the theory of a thermo-microstretch elastic solid and fluids, in which he included microstructural expansions and contractions. Microstretch continuum is a model for Bravais lattice with a basis on the atomic level, and a two-phase dipolar solid with a core on the macroscopic level. For example, composite materials reinforced with chopped elastic fibres, porous media whose pores filled with gas or inviscid liquid, asphalt or other elastic inclusions and "solid-liquid" crystals etc. should be characterizable by microstretch solids. A comprehensive review on the micropolar continuum theory has been given in his book by Eringen [4].

Eringen [5] introduced a continuum theory for micropolar electromagnetic thermoelastic solids. Eringen [6] further extended his theory of thermomicrostretch elastic solids [2] to the electromagnetic interactions and termed it as an electromagnetic theory of microstretch elasticity. He presented constitutive relations and motion equations for isotropic thermomicrostretch elastic solids subjected to electro-magnetic fields. In the absence of magnetic flux vectors, the microstretch thermoelastic continuum will be subjected only to electric fields. We shall call such continuum materials as electro-microstretch thermoelastic solids.

The coupled theory of thermoelasticity has been extended by including the thermal relaxation time in the constitutive equations by Lord and Shulman [7] and Green and Lindsay [9]. These new theories eliminate the paradox of infinite velocity of heat propagation and are termed generalized theories of thermoelasticity. The linear theory of micropolar thermoelasticity was developed by extending the theory of micropolar continua to include the thermal effect by Eringen [10] and Nowacki [11] and is known as the micropolar coupled thermoelasticity. Dost and Tabarrok [12] have presented the generalized micropolar thermal elasticity by using Green-Lindsay theory.

To investigate the boundary value problems of the theory of elasticity and thermoelasticity by potential method, it is necessary to construct a fundamental solution of systems of partial differential equations and to establish their basic properties respectively. Hetnarski [13] was the first to study the fundamental solutions in the classical theory of coupled thermoelasticity. Iesan and Pompei [14] presented the equilibrium theory of microstrech elastic solids. The same author [15, 16] investigated the linear theory of microstretch piezoelectricity and discussed some theorems in the theory of microstretch thermopiezoelectricity. El-Karamany [17] derived the constitutive laws, uniqueness theorem and Hamilton’s principle in linear micropolar thermopiezoelectric/piezoelectric continuum with relaxation time. Svanadze [18] discussed the representations of solutions in the theory of thermoelasticity with microtemperatures for microstretch solids. Some qualitative results for the linear theory of thermo-microstretch elastic solids was investigated by Bofill and Quintanilla [19].The fundamental solutions in the microcontinuum fields theories have been constructed by Svanadze [20-24] and Ciarletta et al. [25]. Recently, Kumar and Kansal [26] studied propagation of plane waves and fundamental solution in the theories of thermoelastic diffusive materials with voids and Sherief et al. [27] derived the fundamental solutions for axi-symmetric translational motion of a microstretch fluid. Useful information on fundamentals solutions of differential equations is also contained in the literature [28-30].

In this article, the propagation of plane waves and fundamental solution in an electro-microstretch generalized thermoelastic solid has been investigated. The phase velocity, attenuation coefficient, specific loss and penetration depth have been computed numerically and depicted graphically. The representation of the fundamental solution of system of equations in the case of steady oscillations is considered in terms of elementary functions and basic properties of the fundamental solution are established.

2. Basic equations

Following Eringen [2] and Iesan and Quintanilla [16], the equations of motion and the constitutive relations in a homogeneous isotropic electro-microstretch thermoelastic solid in the absence of body forces, body couples, stretch force, heat sources and charge density, are given by

$$(\lambda+\mu)\,grad\ div\ \boldsymbol{u}+(\mu+K)\Delta\boldsymbol{u}+K\ curl\ \boldsymbol{\varphi}+\lambda_O\ grad\ \varphi^*-\beta_1\left(1+\tau_1\frac{\partial}{\partial t}\right)grad\ T=\rho\ddot{\boldsymbol{u}}, \qquad (2.1)$$

$$(\gamma\Delta-2K)\boldsymbol{\varphi}+(\alpha+\beta)\,grad\ div\ \boldsymbol{\varphi}+K\ curl\ \boldsymbol{u}=\rho j\ddot{\boldsymbol{\varphi}}, \qquad (2.2)$$

$$(\alpha_0\Delta-\lambda_1)\varphi^*+\nu_1\left(T+\tau_1\dot{T}\right)-\lambda_0\ div.\boldsymbol{u}+\lambda_2\ div.\boldsymbol{E}=\frac{\rho j_0}{2}\ddot{\varphi}^*, \qquad (2.3)$$

$$\beta_1T_0\left(1+\varepsilon\tau_0\frac{\partial}{\partial t}\right)div.\dot{\boldsymbol{u}}+\nu_1T_0\left(1+\varepsilon\tau_0\frac{\partial}{\partial t}\right)\dot{\varphi}^*+\rho C^*\left(1+\tau_0\frac{\partial}{\partial t}\right)\dot{T}=K^*\Delta T, \qquad (2.4)$$

$$D_{i,i}=0\,, \tag{2.5}$$

and constitutive relations are

$$t_{ij}=\lambda u_{r,r}\delta_{ij}+\mu\left(u_{i,j}+u_{j,i}\right)+K\left(u_{j,i}-\varepsilon_{ijr}\varphi_r\right)+\lambda_o\delta_{ij}\varphi^*-\beta_1(1+\tau_1\frac{\partial}{\partial t})T\delta_{ij}, \tag{2.6}$$

$$m_{ij}=\alpha\varphi_{r,r}\delta_{ij}+\beta\varphi_{i,j}+\gamma\varphi_{j,i}+b_0\varepsilon_{mji}\varphi^*_{,m}\,, \tag{2.7}$$

$$\lambda_i^*=\alpha_0\varphi^*_{,i}+b_0\varepsilon_{ijm}\varphi_{j,m}+\lambda_2E_i\,, \tag{2.8}$$

$$D_i=\chi^E E_i-\lambda_3\varepsilon_{rsi}\phi_{s,r}-\lambda_2\phi^*_{,i}, \tag{2.9}$$

where λ, μ are Lame's constants, K,α,β and γ are micropolar constants, $\lambda_o,\lambda_1,\alpha_o,b_o$ are microstretch constants , λ_2,λ_3 are the coupling constants for microstretch and microrotational dielectric effects, respectively, χ^E is the dielectric susceptibility, ρ is the mass density, $\boldsymbol{u}=\left(u_1,u_2,u_3\right)$ is the displacement vector and $\boldsymbol{\varphi}=\left(\varphi_1,\varphi_2,\varphi_3\right)$ is the microrotation vector, φ^* is the microstretch scalar function, T is the temperature measured from constant temperature $T_0\left(T_0\neq 0\right)$, $\left|T/T_0\right|\ll 1$, $\boldsymbol{E}=\left(E_1,E_2,E_3\right)$ is the electric field vector, D_i are the components of dielectric displacement vector, K^* is the coefficient of the thermal conductivity, C^* is the specific heat at constant strain, $\beta_1=\left(3\lambda+2\mu+K\right)\alpha_{t1}$, $\nu_1=\left(3\lambda+2\mu+K\right)\alpha_{t2}$, α_{t1},α_{t2} are coefficients of linear thermal expansion, j is the microintertia, j_o is the microinertia of the microelements, σ_{ij} and m_{ij} are components of stress and couple stress tensors respectively, λ_i^* are the components of microstress tensor, $e_{ij}\left(=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right)\right)$ are components of infinitesimal strain, e_{kk} is the dilatation, δ_{ij} is the Kronecker delta, Δ is the Laplacian operator, τ_0,τ_1 are thermal relaxation times with $\tau_1\geq\tau_0\geq 0$. Here $\tau_0=\tau_1=0$ for Coupled Thermoelasitc (CT) model, $\tau_1=0,\ \varepsilon=1$ for Lord-Shulman (L-S) model and $\tau_1>0,\ \ \varepsilon=0$ for Green-Lindsay (G-L) model.

In the above equations, a comma followed by a suffix denotes spatial derivative and a superposed dot denotes the derivative with respect to time respectively.

For two- dimensional problem, we have

$$\boldsymbol{u}=(u_1,0,u_3)\,,\ \boldsymbol{\varphi}=(0,\varphi_2,0)\,. \tag{2.10}$$

We define the following dimensionless quantities

$$x_i^{'}=\frac{\omega^*}{c_1}x_i,\quad u_i^{'}=\frac{\rho c_1\omega^*}{\beta_1T_o}u_i,\quad T^{'}=\frac{T}{T_o},\quad t^{'}=\omega^*t,\quad \tau_o^{'}=\omega^*\tau_o,\quad \tau^{0'}=\omega^*\tau^0\,,$$

$$\tau_1^{'}=\omega^*\tau_1\,,\quad \tau^{1'}=\omega^*\tau^1,\quad \varphi^{*'}=\frac{\rho c_1^2}{\beta_1T_o}\varphi^*,\quad \varphi_i^{'}=\frac{\rho c_1^2}{\beta_1T_o}\varphi_i,$$

$$E_i^{'}=\frac{\lambda_2\omega^*}{c_1\beta_1T_o}E_i, \tag{2.11}$$

where $\omega^* = \dfrac{\rho C^* c_1^2}{K^*}$, $c_1^2 = \dfrac{\lambda + 2\mu + K}{\rho}$, ω^* is the characteristic frequency of the medium.

Upon introducing the quantities (2.11) in equations (2.1)-(2.5), with the aid of (2.10) and after suppressing the primes, we obtain

$$\delta^2 \frac{\partial e}{\partial x_1} + \left(1-\delta^2\right)\Delta u_1 - a_1 \frac{\partial \varphi_2}{\partial x_3} + a_2 \frac{\partial \varphi^*}{\partial x_1} - \tau_t^1 \frac{\partial T}{\partial x_1} = \frac{\partial^2 u_1}{\partial t^2}, \tag{2.12}$$

$$\delta^2 \frac{\partial e}{\partial x_3} + \left(1-\delta^2\right)\Delta u_3 + a_1 \frac{\partial \varphi_2}{\partial x_1} + a_2 \frac{\partial \varphi^*}{\partial x_3} - \tau_t^1 \frac{\partial T}{\partial x_3} = \frac{\partial^2 u_3}{\partial t^2}, \tag{2.13}$$

$$a_4 \Delta \varphi_2 + a_5 \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) - a_6 \varphi_2 = \frac{\partial^2 \varphi_2}{\partial t^2}, \tag{2.14}$$

$$\left(\delta_1^2 \Delta - a_7\right)\varphi^* - a_8 e + a_9 \tau_t^1 T + a_{18} e_1 = \frac{\partial^2 \varphi^*}{\partial t^2}, \tag{2.15}$$

$$\Delta T = a_{11} \tau_e^0 \frac{\partial e}{\partial t} + a_{12} \tau_e^0 \frac{\partial \varphi^*}{\partial t} + \tau_t^0 \frac{\partial T}{\partial t}, \tag{2.16}$$

$$a_{19} e_1 - \Delta \varphi^* = 0, \tag{2.17}$$

where

$$\left(a_1, a_2\right) = \frac{1}{\rho c_1^2}\left(K, \lambda_0\right), \quad \left(a_4, a_5, a_6\right) = \frac{1}{j\rho}\left(\frac{\gamma}{c_1^2}, \frac{K}{\omega^{*2}}, \frac{2K}{\omega^{*2}} \right), \quad \left(a_7, a_8, a_9\right) = \frac{2}{j_0 \omega^{*2}}\left(\frac{\lambda_1}{\rho}, \frac{\lambda_0}{\rho}, \frac{\nu_1 c_1^2}{\beta_1} \right),$$

$$\left(a_{11}, a_{12}\right) = \frac{1}{K^* \omega^*}\left(\frac{T_0 \beta_1^2}{\rho}, \frac{\beta_1 T_0 \nu_1}{\rho} \right), \quad \delta^2 = \frac{\lambda + \mu}{\rho c_1^2}, \quad \delta_1^2 = \frac{c_2^2}{c_1^2}, \quad c_2^2 = \frac{2\alpha_0}{\rho j_0}, \quad a_{18} = \frac{2 c_1^2}{j_0 \omega^{*2}},$$

$$a_{19} = \frac{\rho c_1^4}{\lambda_2^2 \omega^{*2}} \chi^E, \quad e = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_3}{\partial x_3}, \quad e_1 = \frac{\partial E_1}{\partial x_1} + \frac{\partial E_3}{\partial x_3}, \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}, \quad \text{and} \quad \tau_t^1 = 1 + \tau_1 \frac{\partial}{\partial t},$$

$$\tau_t^0 = 1 + \tau_0 \frac{\partial}{\partial t}, \quad \tau_e^0 = 1 + \varepsilon \tau_0 \frac{\partial}{\partial t}.$$

Introducing the potential functions ϕ, ψ *and* ψ_1 through the relations

$$u_1 = \frac{\partial \phi}{\partial x_1} - \frac{\partial \psi}{\partial x_3}, \quad u_3 = \frac{\partial \phi}{\partial x_3} + \frac{\partial \psi}{\partial x_1}, \quad E_i = \psi_{1,i}, \tag{2.18}$$

in the equations (2.12)-(2.17), we obtain

$$\Delta \phi + a_2 \varphi^* - \tau_t^1 T = \ddot{\phi}, \tag{2.19}$$

$$\left(1-\delta^2\right)\Delta \psi + a_1 \varphi_2 = \ddot{\psi}, \tag{2.20}$$

$$\left(a_4 \Delta - a_6\right)\varphi_2 - a_5 \Delta \psi = \ddot{\varphi}_2, \tag{2.21}$$

$$-a_8 \Delta \phi + \left(\delta_1^2 \Delta - a_7\right)\varphi^* + a_9 \tau_t^1 T + a_{18} \Delta \psi_1 = \ddot{\varphi}^*, \tag{2.22}$$

$$\Delta T = \tau_e^0\left(a_{11}\Delta\dot{\phi} + a_{12}\dot{\varphi}^*\right) + \tau_t^0\dot{T}, \tag{2.23}$$

$$a_{19}\Delta\psi_1 - \Delta\varphi^* = 0. \tag{2.24}$$

3. Solution of plane waves

For plane harmonic waves, we assume the solution of the form

$$(\phi,\psi,\varphi_2,T,\varphi^*,\psi_1) = \left\{\bar{\phi},\bar{\psi},\bar{\varphi}_2,\bar{T},\bar{\varphi}^*,\bar{\psi}_1\right\}\exp[i(\xi(x_1l_1 + x_3l_3) - \omega t)], \tag{3.1}$$

where ω is the frequency and ξ is the wave number. $\bar{\phi}, \bar{\psi}, \bar{\varphi}_2, \bar{T}, \bar{\varphi}^*, \bar{\psi}_1$ are undetermined constants that are independent of time t and coordinates x_m $(m = 1,3)$. l_1 and l_3 are the direction cosines of the wave normal to the x_1x_3 – plane with the property $l_1^2 + l_3^2 = 1$.

Using (3.1) in (2.19)-(2.24), we obtain

$$\left(\omega^2 - \xi^2\right)\bar{\phi} + a_2\bar{\varphi}^* - \tau_1^{"}\bar{T} = 0, \tag{3.2}$$

$$\left(\omega^2 - \left(1-\delta^2\right)\xi^2\right)\bar{\psi} + a_1\bar{\varphi}_2 = 0, \tag{3.3}$$

$$a_5\xi^2\bar{\psi} + \left(\omega^2 - a_4\xi^2 - a_6\right)\bar{\varphi}_2 = 0, \tag{3.4}$$

$$a_8\xi^2\bar{\phi} + \left(\omega^2 - \delta_1^2\xi^2 - a_7\right)\bar{\varphi}^* + a_9\tau_t^{"}\bar{T} - a_{18}\xi^2\bar{\psi}_1 = 0, \tag{3.5}$$

$$-a_{11}\xi^2\tau_e^{10}\bar{\phi} + a_{12}\tau_e^{10}\bar{\varphi}^* + \left(\xi^2 - i\omega - \tau_0\omega^2\right)\bar{T} = 0, \tag{3.6}$$

$$a_{19}\psi_1 - \varphi^* = 0, \tag{3.7}$$

where $\tau_t^{"} = \left(1 - i\omega\tau_1\right)$, $\tau_e^{10} = -i\omega(1 - i\omega\varepsilon\tau_0)$.

The system of equations (3.2) and (3.5)-(3.7) has a non-trivial solution if the determinant of the coefficients $\left[\bar{\phi},\bar{T},\bar{\varphi}^*,\bar{\psi}_1\right]^T$ vanishes, which yields to the following polynomial characteristic equation

$$F_1\xi^6 + F_2\xi^4 + F_3\xi^2 + F_4 = 0, \tag{3.8}$$

where $F_1 = \left(b_{16}\delta_1^2 - a_{18}\right)$, $F_2 = a_{18}\left(\omega^2 + b_{15} + b_{13}\tau_t^{"}\right) - b_{16}\left(\delta_1^2\left(\omega^2 + b_{15}\right) + b_{11} + a_2a_8\right)$,

$$F_3 = -a_{18}b_{15}\omega^2 + b_{16}\begin{bmatrix}\omega^2\left(b_{11} + \delta_1^2 b_{15}\right) + b_{15}\left(b_{11} + a_2a_8\right) \\ +b_{12}\left(b_{14} + a_2b_{13}\right) - \left(a_8b_{14} + b_{13}\delta_1^2\right)\tau_t^{"}\end{bmatrix}, \quad F_4 = b_{16}\left[b_{11}\left(b_{13}\tau_t^{"} - \omega^2 b_{15}\right) - b_{12}b_{14}\omega^2\right],$$

and $b_{11} = \omega^2 - a_7$, $b_{12} = a_9\tau_t^{"}$, $b_{13} = -a_{11}\tau_e^{10}$, $b_{14} = a_{12}\tau_e^{10}$, $b_{15} = i\omega + \tau_0\omega^2$, $b_{16} = -a_{19}$.

Similarly, the system of equations (3.3) and (3.4) has a non-trivial solution if the determinant of the coefficients $\left[\bar{\psi},\bar{\varphi}_2\right]^T$ vanishes, which yields to the following polynomial characteristic equation:

$$F_5\xi^4 + F_6\xi^2 + F_7 = 0, \tag{3.9}$$

where $F_5 = a_4\left(1-\delta^2\right)$, $F_6 = -\left[a_1a_5 + a_4\omega^2 + \left(1-\delta^2\right)\left(\omega^2 - a_6\right)\right]$, $F_7 = \omega^2\left(\omega^2 - a_6\right)$.

Solving (3.8) we obtain six roots of ξ, in which three roots ξ_1, ξ_2 and ξ_3 corresponds to positive x_3-direction and other three roots $-\xi_1, -\xi_2$ and $-\xi_3$ correspond to negative x_3-direction. Now and after, we will restrict our work to positive x_3-direction. Corresponding to roots ξ_1, ξ_2 and ξ_3 there exist three waves in descending order of their velocities, namely LD-wave, T-wave and LM-wave. Likewise, on solving (3.9) we obtain four roots of ξ, in which two roots ξ_4 and ξ_5 correspond to positive x_3-direction represents the two waves in descending order of their velocities, namely CD-I and CD-II waves.

We now derive the expressions for phase velocity, attenuation coefficient, specific loss and penetration depth of these waves.

(i) Phase velocity.

The phase velocities are given by

$$V_i = \frac{\omega}{\left|\mathrm{Re}(\xi_i)\right|};\ \ i = 1,2,3,4,5,$$

where V_1, V_2, V_3, V_4, V_5 are the phase velocities of LD, T ,LM ,CD-I and CD-II waves respectively.

(ii) Attenuation coefficient.

The attenuation coefficients are defined as

$$Q_i = \mathrm{Im}(\xi_i);\ i = 1, 2, 3, 4, 5,$$

where Q_1, Q_2, Q_3, Q_4, Q_5 are the attenuation coefficients of LD, T ,LM ,CD-I and CD-II waves respectively.

(iii) Specific loss.

The specific loss is the ratio of energy ($\overline{W}$) dissipated in taking a specimen through a stress cycle, to the elastic energy (W) stored in the specimen when the strain is maximum. The specific loss is the most direct method of defining internal friction for a material. For a sinusoidal plane wave of small amplitude, Kolsky [32], shows that the specific loss ($\overline{W}/W$) equals 4π times the absolute value of the ratios of imaginary part of ξ to the real part of ξ, i.e.

$$R_i = \left(\frac{\overline{W}}{W}\right)_i = 4\pi\left|\frac{\mathrm{Im}(\xi_i)}{\mathrm{Re}(\xi_i)}\right|;\ \ i = 1,\ 2,\ 3,\ 4, 5.$$

(iv) Penetration depth.

The penetration depths are defined by

$$S_i = \frac{1}{\left|\mathrm{Im}(\xi_i)\right|};\ i = 1,\ 2,\ 3,\ 4, 5.$$

4. Steady oscillations

For steady oscillations, we assume the displacement vector, microrotation, microstretch and, temperature change of the form

$$(\boldsymbol{u}(\boldsymbol{x},t),\boldsymbol{\varphi}(\boldsymbol{x},t),\varphi^*(\boldsymbol{x},t),T(\boldsymbol{x},t))=[(\boldsymbol{u},\boldsymbol{\varphi},\varphi^*,T)e^{-\iota\omega t}], \tag{4.1}$$

Using (2.11) in equations (2.1)-(2.5), after simplification and suppressing the primes, yield

$$\delta^2\Delta.\,\boldsymbol{u}+\left(1-\delta^2\right)grad\ div\ \boldsymbol{u}+a_1\ curl\ \boldsymbol{\varphi}-\beta_1\tau_t^1 grad\ T+a_2\ grad\ \varphi^*=\rho\ddot{\boldsymbol{u}}, \tag{4.2}$$

$$\left(a_4\Delta-a_6\right)\boldsymbol{\varphi}+a_{17}\ grad\ div\,\boldsymbol{\varphi}+a_5\ curl\ \boldsymbol{u}=\ddot{\varphi}, \tag{4.3}$$

$$\left(\left(\delta_1^2+\delta_2^2\right)\Delta-a_7\right)\varphi^*-a_8\ div\ \boldsymbol{u}+a_9\tau_t^1 T=\ddot{\varphi}^*, \tag{4.4}$$

$$\tau_e^0\left(a_{11}div\ \dot{\boldsymbol{u}}+a_{12}\dot{\varphi}^*\right)+\tau_t^0\dot{T}=\Delta T, \tag{4.5}$$

where

$$a_{17}=\frac{\alpha+\beta}{j\rho c_1^2},\delta_2^2=\frac{2\lambda_2^2}{j_0\rho c_1^2\chi^E} \tag{4.6}$$

Using (4.1) into (4.2)-(4.5), we obtain the system of equations of steady oscillations as

$$\left(\delta^2\Delta+\omega^2\right)\boldsymbol{u}+\left(1-\delta^2\right)grad\ div\ \boldsymbol{u}+a_1\ curl\ \boldsymbol{\varphi}+a_2\ grad\ \varphi^*-\beta_1\tau^* grad\ T=0,$$

$$\left(a_4\Delta+\mu^*\right)\boldsymbol{\varphi}+a_{17}\ grad\ div\,\boldsymbol{\varphi}+a_5\ curl\ \boldsymbol{u}=0,$$

$$\left(\left(\delta_1^2+\delta_2^2\right)\Delta+\zeta^*\right)\varphi^*-a_8\ div\ \boldsymbol{u}+a_9\tau^* T=0,$$

$$-\tau_\varepsilon^{10}\left(a_{11}div\ \boldsymbol{u}+a_{12}\varphi^*\right)+\left(-\tau_t^{10}+\Delta\right)T=0, \tag{4.7}$$

where

$$\tau^*=\left(1-i\omega\tau_1\right),\ \mu^*=\left(\omega^2-a_6\right),\ \zeta^*=\left(\omega^2-a_7\right),\ \tau_\varepsilon^{10}=-i\omega\left(1-i\varepsilon\tau_0\omega\right),\ \tau_t^{10}=-i\omega\left(1-i\omega\tau_0\right). \tag{4.8}$$

We introduce the matrix differential operator

$$F(D_x)=\left\|F_{gh}(D_x)\right\|_{8\times8}, \tag{4.9}$$

where

$$F_{mn}(D_x)=\left(\delta^2\Delta+\omega^2\right)\delta_{mn}+\left(1-\delta^2\right)\frac{\partial^2}{\partial x_m\partial x_n},\ F_{m,n+3}(D_x)=a_1\sum_{r=1}^{3}\varepsilon_{mrn}\frac{\partial}{\partial x_r},\ F_{m7}(D_x)=a_2\frac{\partial}{\partial x_m},$$

$$F_{m8}(D_x)=-\beta_1\tau^*\frac{\partial}{\partial x_m},\ F_{m+3,n+3}(D_x)=\left(a_4\Delta+\mu^*\right)\delta_{mn}+a_{17}\frac{\partial^2}{\partial x_m\partial x_n},\ F_{m+3,n}(D_x)=a_5\sum_{r=1}^{3}\varepsilon_{mrn}\frac{\partial}{\partial x_r},$$

$$F_{m+3,7}(D_x)=F_{7,n+3}=F_{m+3,8}(D_x)=F_{8,n+3}=0,\ F_{77}(D_x)=\left(\delta_1^2+\delta_2^2\right)\Delta+\zeta^*,\ F_{78}(D_x)=a_9\tau^*,\ F_{87}(D_x)=-a_{12}\tau_\varepsilon^{10},$$

$$F_{88}(D_x)=\left(-\tau_t^{10}+\Delta\right),\ F_{7n}(D_x)=-a_8\frac{\partial}{\partial x_n},\ F_{8n}(D_x)=-a_{11}\tau_\varepsilon^{10}\frac{\partial}{\partial x_n},\ m,n=1,2,3\ . \tag{4.10}$$

Here ε_{mrn} is the alternating tensor and δ_{mn} is the Kronecker delta function.
The system of equations (4.7) can be written as

$$F(D_x)U(x)=0, \tag{4.11}$$

where $U=\left(\boldsymbol{u},\boldsymbol{\varphi},\varphi^*,T\right)$ is a eight component vector function on E^3.

Definition: The fundamental solution of the system of equations (4.7) (the fundamental matrix of operator F) is the matrix $G(x)=\left\|G_{gh}(x)\right\|_{8\times 8}$, satisfying condition [28].

$$F(D_x)G(x)=\delta(x)I(x), \tag{4.12}$$

where δ is the Dirac delta, $I=\left\|\delta_{gh}\right\|_{8\times 8}$ is the unit matrix and $x\in E^3$.
Now we construct $G(x)$ in terms of elementary functions.

5. Fundamental solution of system of equation of steady oscillations
We consider the system of equations

$$\delta^2\Delta\boldsymbol{u}+\left(1-\delta^2\right)grad\ div\ \boldsymbol{u}+a_5\ curl\ \boldsymbol{\varphi}-a_8\ grad\ \varphi^*-a_{11}\tau_\varepsilon^{10}grad\ T+\omega^2\boldsymbol{u}=\boldsymbol{H'}, \tag{5.1}$$

$$a_1\ curl\ \boldsymbol{u}+\left(a_4\Delta+\mu^*\right)\boldsymbol{\varphi}+a_{17}\ grad\ div\ \boldsymbol{\varphi}=\boldsymbol{H''}, \tag{5.2}$$

$$a_2 div\ \boldsymbol{u}+\left(\left(\delta_1^2+\delta_2^2\right)\Delta+\zeta^*\right)\varphi^*-a_{12}\tau_\varepsilon^{10}T=Z, \tag{5.3}$$

$$-\beta_1\tau^* div\ \boldsymbol{u}+a_9\tau^*\varphi^*+\left(-\tau_t^{10}+\Delta\right)T=L, \tag{5.4}$$

where $\boldsymbol{H'},\boldsymbol{H''}$ are three component vector function on E^3 and Z,L are scalar functions on E^3.

The system of equations (5.1)-(5.4) may be written in the form

$$F^{tr}(D_x)U(x)=Q(x), \tag{5.5}$$

where F^{tr} is the transpose of matrix F, $Q=(\boldsymbol{H'},\boldsymbol{H''},\ Z,L)$ and $x\in E^3$.

Applying the operator *div* to (5.1) and (5.2), we obtain

$$\left(\Delta+\omega^2\right)div\ \boldsymbol{u}-a_8\ \Delta\ \varphi^*-a_{11}\tau_\varepsilon^{10}\Delta\ T=div\ \boldsymbol{H'},$$

$$\left(v^*\Delta+\mu^*\right)div\ \boldsymbol{\varphi}=div\ \boldsymbol{H''},$$

$$a_2 div\ \boldsymbol{u}+\left(\left(\delta_1^2+\delta_2^2\right)\Delta+\zeta^*\right)\varphi^*-a_{12}\tau_\varepsilon^{10}T=Z,$$

$$-\beta_1\tau^* div\ \boldsymbol{u}+a_9\tau^*\varphi^*+\left(-\tau_t^{10}+\Delta\right)T=L, \tag{5.6}$$

where $v^*=a_4\Delta+a_{17}$.

Equations $(5.6)_1$, $(5.6)_3$ and $(5.6)_4$ can be written in the form:

$$N(\Delta)S = \bar{Q}, \tag{5.7}$$

where $S = (div\, \boldsymbol{u}, \varphi^*, T)$, $\bar{Q} = (d_1, d_2, d_3) = (div\, \boldsymbol{H'}, Z, L)$, and

$$N(\Delta) = \|N_{mn}(\Delta)\|_{3\times 3} = \begin{Vmatrix} \Delta + \omega^2 & -a_8\,\Delta & -a_{11}\tau_\varepsilon^{10}\Delta \\ a_2 & (\delta_1^2 + \delta_2^2)\Delta + \zeta^* & -a_{12}\tau_\varepsilon^{10} \\ -\beta_1\tau^* & a_9\tau^* & -\tau_t^{10} + \Delta \end{Vmatrix}_{3\times 3} \tag{5.8}$$

Equations $(5.6)_1$, $(5.6)_3$ and $(5.6)_4$ can be written as

$$\Gamma_1(\Delta)S = \boldsymbol{\Psi} \tag{5.9}$$

where

$$\boldsymbol{\Psi} = (\Psi_1, \Psi_2, \Psi_3),\ ,\ \Psi_n = e^*\sum_{m=1}^{3} N_{mn}^* d_m\ ,$$

$$\Gamma_1(\Delta) = e^* \det N(\Delta),\ \ e^* = 1/(\delta_1^2 + \delta_2^2),\ \ n = 1, 2, 3, \tag{5.10}$$

and N_{mn}^* is the cofactor of the elements of the matrix N.
From (5.8) and (5.10), we see that

$$\Gamma_1(\Delta) = \prod_{m=1}^{3}(\Delta + \lambda_m^2), \tag{5.11}$$

where λ_m^2, $m = 1,\ 2,\ 3$ are the roots of the equation $\Gamma_1(-k) = 0$ (with respect to k).
From equation $(5.6)_2$, it follows that

$$(\Delta + \lambda_6^2)div\,\boldsymbol{\varphi} = \frac{1}{v^*}div\boldsymbol{H''}, \tag{5.12}$$

where $\lambda_6^2 = \mu^*/v^*$.

Applying the operators $a_4\Delta + \mu^*$ and $a_5\ curl$ to (5.1) and (5.2), respectively, we obtain

$$(a_4\Delta + \mu^*)\left(\delta^2\Delta\boldsymbol{u} + (1-\delta^2)grad\ div\boldsymbol{u} + \omega^2\vec{u}\right) + a_5(a_4\Delta + \mu^*)curl\boldsymbol{\varphi} =$$

$$= (a_4\Delta + \mu^*)\left(\boldsymbol{H'} + a_8\ grad\varphi^* + a_{11}\tau_\varepsilon^{10} gradT\right), \tag{5.13}$$

and $$a_5(a_4\Delta + \mu^*)curl\boldsymbol{\varphi} = -a_5 a_1\ curl\ curl\ \boldsymbol{u} + a_5\ curl\ \boldsymbol{H''}. \tag{5.14}$$

Now

$$curl\ curl\ \boldsymbol{u} = grad\ div\ \boldsymbol{u} - \Delta\boldsymbol{u}\ , \tag{5.15}$$

Using (5.14) and (5.15) in (5.13), we obtain

$$(a_4\Delta+\mu^*)\left(\delta^2\Delta\boldsymbol{u}+\left(1-\delta^2\right)grad\ div\ \boldsymbol{u}+\omega^2\boldsymbol{u}\right)+a_5a_1\Delta\boldsymbol{u}-a_5a_1\ grad\ div\ \boldsymbol{u}=$$

$$=(a_4\Delta+\mu^*)\left(\boldsymbol{H'}+a_8\ grad\varphi^*+a_{11}\tau_\varepsilon^{10}gradT\right)-a_5\ curl\ \boldsymbol{H''}. \tag{5.16}$$

The above equation can also be written as

$$\left\{\left[(a_4\Delta+\mu^*)\delta^2+a_5a_1\right]\Delta+(a_4\Delta+\mu^*)\omega^2\right\}\boldsymbol{u}$$

$$=-\left[\left(1-\delta^2\right)(a_4\Delta+\mu^*)-a_5a_1\right]grad\ div\ \boldsymbol{u}+$$

$$+(a_4\Delta+\mu^*)\left[\boldsymbol{H'}+a_8\ grad\varphi^*+a_{11}\tau_\varepsilon^{10}gradT\right]-a_5\ curl\ \boldsymbol{H''}\ . \tag{5.17}$$

Applying the operator $\Gamma_1(\Delta)$ to (5.17) and using (5.9), we obtain

$$\Gamma_1(\Delta)\left[a_4\delta^2\Delta^2+\left(\mu^*\delta^2+a_4\omega^2+a_5a_1\right)\Delta+\mu^*\omega^2\right]\boldsymbol{u}=$$

$$=-\left[\left(1-\delta^2\right)(a_4\Delta+\mu^*)-a_5a_1\right]grad\ \Psi_1+$$

$$+(a_4\Delta+\mu^*)\left[\Gamma_1(\Delta)\boldsymbol{H'}+a_8\ grad\ \Psi_2+a_{11}\tau_\varepsilon^{10}grad\ \Psi_3\right]-a_5\ \Gamma_1(\Delta)\ curl\ \boldsymbol{H''}. \tag{5.18}$$

The above equation can be written as

$$\Gamma_1(\Delta)\Gamma_2(\Delta)\vec{u}=\boldsymbol{\Psi}', \tag{5.19}$$

where $\Gamma_2(\Delta)=f^*\det\begin{Vmatrix}\delta^2\Delta+\omega^2 & a_5\Delta\\ -a_1 & a_4\Delta+\mu^*\end{Vmatrix},\quad f^*=\dfrac{1}{\delta^2a_4},$ (5.20)

and $\boldsymbol{\Psi}'=f^*\begin{Bmatrix}-\left[\left(1-\delta^2\right)(a_4\Delta+\mu^*)-a_5a_1\right]grad\ \Psi_1\\ +(a_4\Delta+\mu^*)\left[\Gamma_1(\Delta)\boldsymbol{H'}+a_8\ grad\ \Psi_2+a_{11}\tau_\varepsilon^{10}grad\ \Psi_3\right]\\ -a_5\ \Gamma_1(\Delta)\ curl\ \boldsymbol{H''}\end{Bmatrix}.$ (5.21)

It can be seen that

$$\Gamma_2(\Delta)=(\Delta+\lambda_4^2)(\Delta+\lambda_5^2), \tag{5.22}$$

where $\lambda_4^2,\ \lambda_5^2$ are the roots of the equation $\Gamma_2(-k)=0$ (with respect to k).

Applying the operators $a_1\ curl$ and $(\delta^2\Delta+\omega^2)$ to (5.1) and (5.2), respectively, we obtain

$$a_1(\delta^2\Delta+\omega^2)curl\ \boldsymbol{u}=a_1curl\ \boldsymbol{H'}-a_1a_5\ curl\ curl\ \boldsymbol{\varphi}, \tag{5.23}$$

$$(\delta^2\Delta+\omega^2)\left(a_4\Delta+\mu^*\right)\boldsymbol{\varphi}+a_{17}\ (\delta^2\Delta+\omega^2)grad\ div\ \boldsymbol{\varphi}+$$

$$+a_1(\delta^2\Delta+\omega^2)\ curl\ \boldsymbol{u}=(\delta^2\Delta+\omega^2)\boldsymbol{H''}. \tag{5.24}$$

Now

$$curl\ curl\ \boldsymbol{\varphi} = grad\ div\ \boldsymbol{\varphi} - \Delta\boldsymbol{\varphi}\,. \qquad (5.25)$$

Using (5.23) and (5.25) in (5.24), we obtain

$$(\delta^2\Delta+\omega^2)\left(a_4\Delta+\mu^*\right)\boldsymbol{\varphi} + a_{17}\ (\delta^2\Delta+\omega^2) grad\ div\ \boldsymbol{\varphi} +$$

$$+ a_1 a_5 \Delta\boldsymbol{\varphi} - a_1 a_5 grad\ div\ \boldsymbol{\varphi} = (\delta^2\Delta+\omega^2)\boldsymbol{H''} - a_1 curl\ \boldsymbol{H'}\,. \qquad (5.26)$$

The above equation may also be written as

$$\left\{\left[\left(a_4\Delta+\mu^*\right)\delta^2 + a_1 a_5\right]\Delta + \left(a_4\Delta+\mu^*\right)\omega^2\right\}\boldsymbol{\varphi} =$$

$$= -\left[a_{17}\ (\delta^2\Delta+\omega^2) - a_1 a_5\right] grad\ div\ \boldsymbol{\varphi} + (\delta^2\Delta+\omega^2)\boldsymbol{H''} - a_1 curl\ \boldsymbol{H'}\,. \qquad (5.27)$$

Applying operator $(\Delta+\lambda_6^2)$ to (5.27) and using (5.12), we obtain

$$(\Delta+\lambda_6^2)\left[a_4\delta^2\Delta^2 + \left(\mu^*\delta^2 + a_4\omega^2 + a_5 a_1\right)\Delta + \mu^*\omega^2\right]\boldsymbol{\varphi} =$$

$$= -a_1(\Delta+\lambda_6^2) curl\ \boldsymbol{H'} + (\Delta+\lambda_6^2)(\delta^2\Delta+\omega^2)\boldsymbol{H''} -$$

$$-\left(1/v^*\right)\left[a_{17}\ (\delta^2\Delta+\omega^2) - a_1 a_5\right] grad\ div \boldsymbol{H''}\,. \qquad (5.28)$$

The above equation may also be rewritten in the form

$$\Gamma_2(\Delta)(\Delta+\lambda_6^2)\vec{\varphi} = \boldsymbol{\Psi}'', \qquad (5.29)$$

where

$$\boldsymbol{\Psi}'' = f^*\begin{Bmatrix} -a_1(\Delta+\lambda_6^2) curl\ \boldsymbol{H'} + (\Delta+\lambda_6^2)(\delta^2\Delta+\omega^2)\boldsymbol{H''} \\ -\left(1/v^*\right)\left[a_{17}\ (\delta^2\Delta+\omega^2) - a_1 a_5\right] grad\ div \boldsymbol{H''} \end{Bmatrix}. \qquad (5.30)$$

From (5.9), (5.19), and (5.29), we obtain,

$$\Theta(\Delta)U(x) = \hat{\boldsymbol{\Psi}}(x), \qquad (5.31)$$

where

$$\hat{\boldsymbol{\Psi}} = \left(\boldsymbol{\Psi}', \boldsymbol{\Psi}'', \Psi_2, \Psi_3\right),$$

$$\Theta(\Delta) = \left\|\Theta_{gh}(\Delta)\right\|_{8\times 8},$$

$$\Theta_{mm}(\Delta) = \Gamma_1(\Delta)\Gamma_2(\Delta) = \prod_{q=1}^{5}(\Delta+\lambda_q^2),\ \ \Theta_{m+3,n+3}(\Delta) = \Gamma_2(\Delta)(\Delta+\lambda_6^2) = \prod_{q=4}^{6}(\Delta+\lambda_q^2), \qquad (5.32)$$

$$\Theta_{77}(\Delta) = \Theta_{88}(\Delta) = \Gamma_1(\Delta),\ \ \Theta_{gh}(\Delta) = 0,\ \ \ m = 1,\ 2,\ 3,\ \ \ g,h = 1,\ 2,\ 3,\ldots,8,\ \ g \neq h\,.$$

Equations (5.10), (5.21) and (5.30) can be rewritten in the form

$$\mathbf{\Psi}' = \left[f^*(a_4\Delta+\mu^*)\Gamma_1(\Delta)J + q_{11}(\Delta) grad\ div \right]\boldsymbol{H'} +$$

$$+ q_{21}(\Delta) curl\ \boldsymbol{H''} + q_{31}(\Delta) grad\ Z + q_{41}(\Delta) grad\ L,$$

$$\mathbf{\Psi}'' = q_{12}(\Delta) curl\ \boldsymbol{H'} + f^* \left\{ (\Delta+\lambda_6^2)(\delta^2\Delta+\omega^2)J + q_{22}(\Delta) grad\ div \right\} \boldsymbol{H''}, \tag{5.33}$$

$$\Psi_2 = q_{13}(\Delta) div\ \boldsymbol{H'} + q_{33}(\Delta)Z + q_{43}(\Delta)L,$$

$$\Psi_3 = q_{14}(\Delta) div\ \boldsymbol{H'} + q_{34}(\Delta)Z + q_{44}(\Delta)L,$$

where $J = \left\| \delta_{gh} \right\|_{3\times 3}$, is the unit matrix.

In (5.33), we have used the following notations:

$$q_{m1}(\Delta) = f^* e^* \left\{ (a_4\Delta+\mu^*)\left[a_8 N_{m2}^* + a_{11}\tau_\varepsilon^{10} N_{m3}^* \right] - \left[\left(1-\delta^2\right)(a_4\Delta+\mu^*) - a_5 a_1 \right] N_{m1}^* \right\},$$

$$q_{21}(\Delta) = -f^* a_1 \Gamma_1(\Delta),\ \ q_{12}(\Delta) = -f^* a_1(\Delta+\lambda_6^2), \quad q_{22}(\Delta) = -\left(f^*/v^*\right)\left[a_{17}\ (\delta^2\Delta+\omega^2) - a_1 a_5 \right],$$

$$q_{1p}(\Delta) = e^* N_{1,p-1}^*,\ \ (p=3,4),\ \ q_{rs}(\Delta) = e^* N_{r-1,s-1}^*,\ \ (r,s=3,4)\ , m=1,3,4. \tag{5.34}$$

Now from equations (5.33), we have

$$\hat{\mathbf{\Psi}}(x) = R^{tr}(D_x)Q(x), \tag{5.35}$$

where R^{tr} is the transpose of matrix R and

$$R = \left\| R_{mn} \right\|_{8\times 8},$$

$$R_{mn}(D_x) = f^*(a_4\Delta+\mu^*)\Gamma_1(\Delta)\delta_{mn} + q_{11}(\Delta)\frac{\partial^2}{\partial x_m \partial x_n},\ \ R_{m,n+3}(D_x) = q_{12}(\Delta)\sum_{r=1}^{3}\varepsilon_{mrn}\frac{\partial}{\partial x_r},$$

$$R_{mp}(D_x) = q_{1,p-4}(\Delta)\frac{\partial}{\partial x_m},\ \ R_{m+3,n}(D_x) = q_{21}(\Delta)\sum_{r=1}^{3}\varepsilon_{mrn}\frac{\partial}{\partial x_r},\ \ R_{pn}(D_x) = q_{p-4,1}(\Delta)\frac{\partial}{\partial x_n},$$

$$R_{m+3,n+3}(D_x) = f^*(\Delta+\lambda_6{}^2)(\delta^2\Delta+\omega^2)\delta_{mn} + q_{22}(\Delta)\frac{\partial^2}{\partial x_m \partial x_n},\ \ R_{p,s}(D_x) = q_{p-4,s-4}(\Delta),$$

$$R_{m+3,7}(D_x) = R_{m+3,p}(D_x) = R_{p,m+3}(D_x) = 0, \quad m,n=1,2,3,\ \ p,s=7,8. \tag{5.36}$$

From (5.5), (5.31) and (5.35), we obtain

$$\Theta U = R^{tr}F^{tr}U\ . \tag{5.37}$$

It implies that

$$R^{tr}F^{tr} = \Theta,$$

and hence

$$F(D_x)R(D_x)=\Theta(\Delta)\,. \tag{5.38}$$

We assume that

$$\lambda_m^2\neq\lambda_n^2\neq 0,\quad m,n=1,2,3,4,5,6\quad m\neq n\,. \tag{5.39}$$

Let

$$Y(x)=\left\|Y_{rs}(x)\right\|_{8\times 8},\ \ Y_{mm}(x)=\sum_{n=1}^{5}r_{1n}\varsigma_n(x), \tag{5.40}$$

$$Y_{m+3,m+3}(x)=\sum_{n=4}^{6}r_{2n}\varsigma_n(x),\ \ Y_{77}(x)=Y_{88}(x)=\sum_{n=1}^{3}r_{3n}\varsigma_n(x),$$

$$Y_{vw}=0,\quad v,w=1,2,.............,8,\ \ v\neq w,\ \ m=1,2,3,$$

where

$$\varsigma_n(x)=-\frac{1}{4\pi|x|}\exp(i\lambda_n|x|),\quad n=1,2,...,6$$

$$r_{1\ell}=\prod_{m=1,m\neq\ell}^{5}(\lambda_m^2-\lambda_\ell^2)^{-1},\quad \ell=1,\ 2,\ 3,\ 4,5\,,$$

$$r_{2v}=\prod_{m=4,m\neq v}^{6}(\lambda_m^2-\lambda_v^2)^{-1},\quad v=4,\ 5,\ 6\,,$$

$$r_{3w}=\prod_{m=1,m\neq w}^{3}(\lambda_m^2-\lambda_w^2)^{-1},\quad w=1,\ 2,3 \tag{5.41}$$

We will prove the following Lemma:

Lemma: The matrix Y defined above is the fundamental matrix of operator $\Theta(\Delta)$, that is

$$\Theta(\Delta)Y(x)=\delta(x)I(x)\,. \tag{5.42}$$

Proof: To prove the lemma, it is sufficient to prove that

$$\Gamma_1(\Delta)\Gamma_2(\Delta)Y_{11}(x)=\delta(x),\quad \Gamma_2(\Delta)(\Delta+\lambda_6^2)Y_{33}(x)=\delta(x),\quad \Gamma_1(\Delta)Y_{66}(x)=\delta(x), \tag{5.43}$$

we find that

$$r_{11}+r_{12}+r_{13}+r_{14}+r_{15}=0,\quad \sum_{j=2}^{5}r_{1j}(\lambda_1^2-\lambda_j^2)=0,\quad \sum_{j=3}^{5}r_{1j}\prod_{m=1}^{2}(\lambda_m^2-\lambda_j^2)=0,$$

$$\sum_{j=4}^{5}r_{1j}\prod_{m=1}^{3}(\lambda_m^2-\lambda_j^2)=0,\quad r_{15}\prod_{m=1}^{4}(\lambda_m^2-\lambda_5^2)=1,$$

$$(\Delta+\lambda_m^2)\varsigma_n(x)=\delta(x)+(\lambda_m^2-\lambda_n^2)\varsigma_n(x),\ \ m,n=1,2,3,4,5\,. \tag{5.44}$$

Now consider

$$\Gamma_1(\Delta)\Gamma_2(\Delta)Y_{11}(x) = (\Delta+\lambda_2^2)(\Delta+\lambda_3^2)(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)\sum_{n=1}^{5} r_{1n}[\delta(x)+(\lambda_1^2-\lambda_n^2)\varsigma_n(x)],$$

$$= (\Delta+\lambda_2^2)(\Delta+\lambda_3^2)(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)\sum_{n=2}^{5} r_{1n}(\lambda_1^2-\lambda_n^2)\varsigma_n(x),$$

$$= (\Delta+\lambda_3^2)(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)\sum_{n=2}^{5} r_{1n}(\lambda_1^2-\lambda_n^2)[\delta(x)+(\lambda_2^2-\lambda_n^2)\varsigma_n(x)],$$

$$= (\Delta+\lambda_3^2)(\Delta+\lambda_4^2)(\Delta+\lambda_5^2)\sum_{n=3}^{5} r_{1n}(\lambda_1^2-\lambda_n^2)(\lambda_2^2-\lambda_n^2)\varsigma_n(x),$$

$$= (\Delta+\lambda_4^2)(\Delta+\lambda_5^2)\sum_{n=3}^{5} r_{1n}(\lambda_1^2-\lambda_n^2)(\lambda_2^2-\lambda_n^2)[\delta(x)+(\lambda_3^2-\lambda_n^2)\varsigma_n(x)],$$

$$= (\Delta+\lambda_5^2)\sum_{n=4}^{5} r_{1n}(\lambda_1^2-\lambda_n^2)(\lambda_2^2-\lambda_n^2)(\lambda_3^2-\lambda_n^2)(\lambda_4^2-\lambda_n^2)[\delta(x)+(\lambda_4^2-\lambda_n^2)\varsigma_n(x)],$$

$$= (\Delta+\lambda_5^2)\varsigma_5(x) = \delta(x) \tag{5.45}$$

Similarly, $(5.43)_2$ and $(5.43)_3$ can be proved.
We introduce the matrix

$$G(x) = R(D_x)Y(x) \tag{5.46}$$

From (5.38), (5.42) and (5.46), we obtain

$$F(D_x)G(x) = F(D_x)R(D_x)Y(x) = \delta(x)I(x)\,. \tag{5.47}$$

Hence, $G(x)$ is a solution to (4.12).
Therefore we have proved the following theorem.

Theorem: The matrix $G(x)$ defined by (5.46) is the fundamental solution of system of equations (4.7).

6. Basic Properties of the matrix $G(x)$

Property 1. Each column of the matrix $G(x)$ is the solution of the system of equations (4.7) at every point $x \in E^3$ except at the origin.

Property 2. The matrix $G(x)$ can be written in the form

$$G = \left\| G_{gh} \right\|_{8\times 8},$$

$$G_{mn}(x) = R_{mn}(D_x)Y_{11}(x),\; G_{m,n+3}(x) = R_{m,n+3}(D_x)Y_{33}(x),\; G_{mp}(x) = R_{mp}(D_x)Y_{66}(x),$$

$$m = 1,\ 2,\dots,\ 8,\quad n = 1,\ 2,\ 3,\quad p = 7, 8 \tag{6.1}$$

7. Particular cases

(i) If we neglect the electro effect, we obtain the same result for fundamental solution as discussed by Svanadze and De Cicco [31] by changing the dimensionless quantities in to physical quantities in case of microstretch generalized thermoelastic solid.

(ii) If we neglect the electrothermal effects, we obtain the same result for fundamental solution as discussed by Svanadze [22] by changing the dimensionless quantities in to physical quantities in case of microstretch elastic solid.

8. Numerical results and discussion

With the view of illustrating theoretical results obtained in the preceding sections and compare these in the context of various theories of electro-microstretch generalized thermoelastic solid. The analysis is conducted for a magnesium crystal-like material. Following [8], the values of physical constants are

$\lambda = 9.4\times10^{10}$ Nm^{-2}, $\mu = 4.0\times10^{10}$ Nm^{-2}, $K = 1.0\times10^{10}\, Nm^{-2}$, $\rho = 1.74\times10^{3}\, Kgm^{-3}$,

$j = 0.2\times10^{-19} m^2$, $\gamma = 0.779\times10^{-9} N$, $\lambda_2 = 1.7\times10^{-4}\, Cm^{-1}$, $\chi^E = 318000000\times10^{-6}\, C^2m^{-2}N^{-1}$.

Thermal and electro parameters are given by

$C^* = 1.04\times10^{3} JKg^{-1}K^{-1}$, $K^* = 1700000\, Jm^{-1}s^{-1}K^{-1}$, $T_0 = 0.000298\times10^{6}$ K,

$\alpha_{t1} = 2.33\times10^{-5}$K^{-1}, $\alpha_{t2} = 2.48\times10^{-5}$K^{-1}, $\tau_1 = 0.01$, $\tau_0 = 0.02$,

and, the microstretch parameters are taken as

$j_o = 0.19\times10^{-19} m^2$, $\alpha_o = 0.779\times10^{-9} N$, $\lambda_o = 0.5\times10^{10} Nm^{-2}$, $\lambda_1 = 0.5\times10^{10} Nm^{-2}$.

The software Matlab 7.0.4 has been used to determine the values of phase velocity, attenuation coefficient, specific loss and penetration depth of plane waves, i.e. LD, T, LM, CD-I and CD-II. The variations of phase velocity, attenuation coefficients, specific loss and penetration depth with respect to frequency have been shown in Figs.1-5, 6-8, 9-11 and 12-14 respectively. In all the Figures, LSE and LSWE corresponds to L-S theory of electro-microstretch generalized thermoelastic solid and microstretch generalized thermoelastic respectively, GLE and GLWE corresponds to G-L theory of electro-microstretch generalized thermoelastic solid and microstretch generalized thermoelastic respectively.

8.1. Phase velocity. From Fig.1, it is noticed that the values of V_1 first increase and then shows a constant behavior. Due to presence of electro effect, the values for LSE and GLE are slightly lower in comparison with LSWE and GLWE respectively.

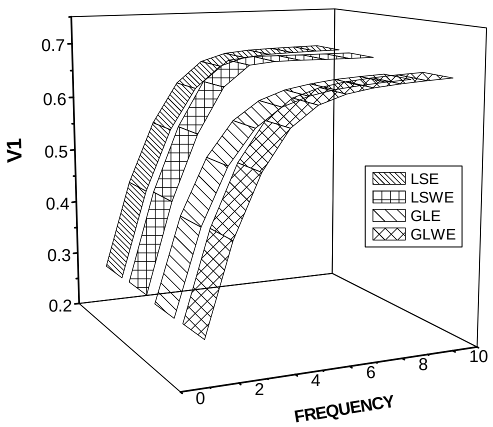

Fig. 1. Variation of phase velocity V_1 with frequency ω.

Figure 2 indicates the trend of variation of V_2 first increase and then decrease rapidly within the range $0 \le \omega \le 4$ and then becomes constant for the range $4 < \omega \le 10$. The values for LSE and LSWE theories are higher than GLE and GLWE theories respectively. Initially for the small values of frequency, the phase velocity V_3 is constant and then increase smoothly for higher values of ω as shown in Fig. 3. In the range $2 \le \omega \le 10$, V_3 attains higher values for both GLE and GLWE theories. Figure 4 depicts that the values of V_4 decrease rapidly and then attains the constant values for all the theories. Figure 5 indicates that the trend of variation and behavior of V_5 is similar as V_4 although the magnitude values are different.

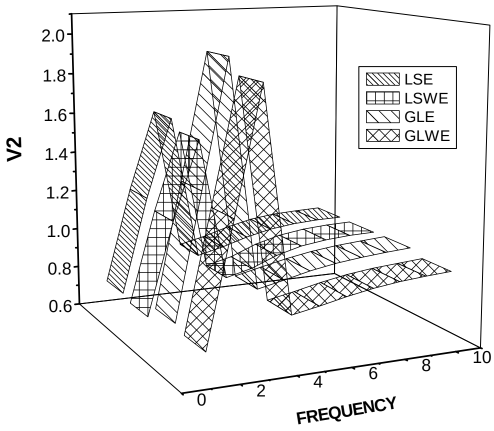

Fig. 2. Variation of phase velocity V_2 with frequency ω.

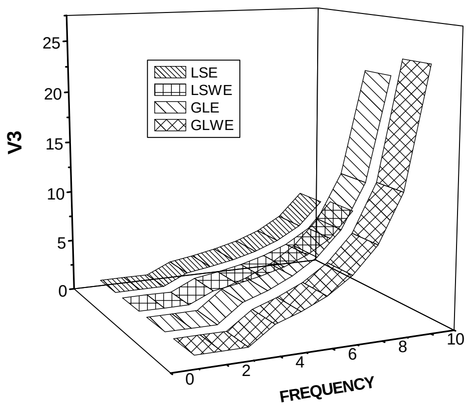

Fig. 3. Variation of phase velocity V_3 with frequency ω.

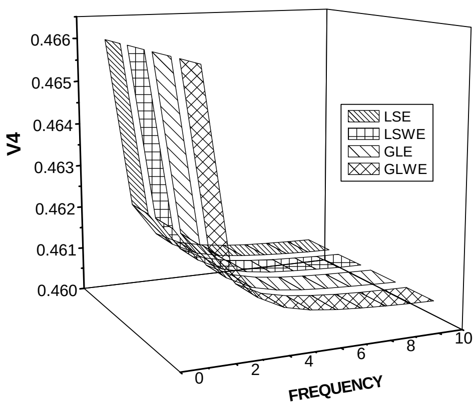

Fig. 4. Variation of phase velocity V_4 with frequency ω.

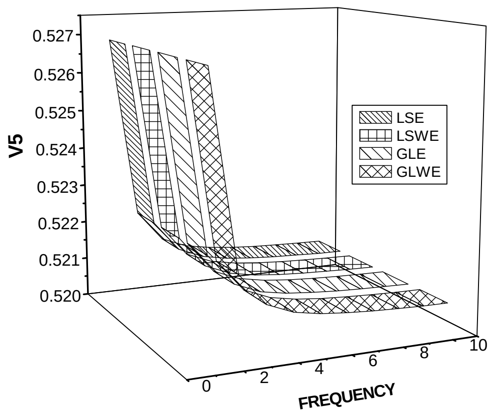

Fig. 5. Variation of phase velocity V_5 with frequency ω.

8.2. Attenuation coefficient. Figure 6 shows that the values of Q_1 increase for smaller values of ω but for higher values of ω the values of Q_1 decrease. Due to presence of electro effect, the values of Q_1 for LSE and GLE are higher in comparison with LSWE and GLWE respectively. It is evident that from Fig. 7 that the values of Q_2 decrease for initial value of ω and then increase and finally become dispersionless. The values of Q_2 for LSWE and GLWE are more as compared to values for LSE and GLE theories respectively. Figure 8 shows that the values of Q_3 decrease for initial values of ω but for higher values of ω the values of Q_3 increase.

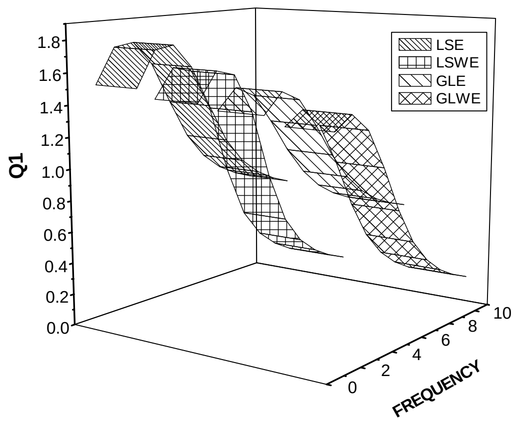

Fig. 6. Variation of attenuation coefficient Q_1 with frequency ω.

8.3. Specific loss. Figure 9 shows that the values of specific loss R_1 decrease monotonically and become constant for higher values of ω. It is noticed from Fig. 10 the values of R_2 decrease for smaller values of ω but higher values of ω the values of R_2 becomes dispersionless. The values of R_2 are higher in case of GLE and GLWE as compared to LSE and LSWE respectivley. Figure 11 incidates that for the small values of ω, the trend of variation and behavior of R_3 is similar for all cases, but for higher values of ω, the values of R_3 for GLE and GLWE increase monotonically whereas for LSE and LSWE remains stationary.

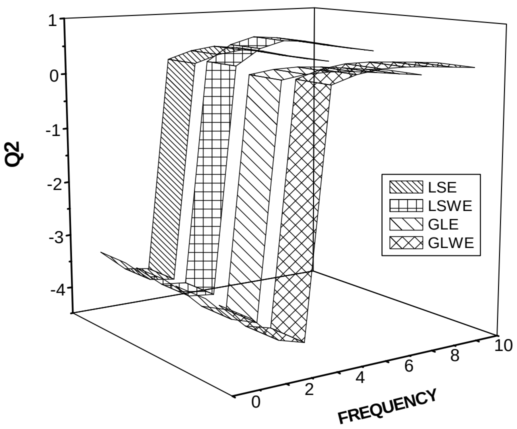

Fig. 7. Variation of attenuation coefficient Q_2 with frequency ω.

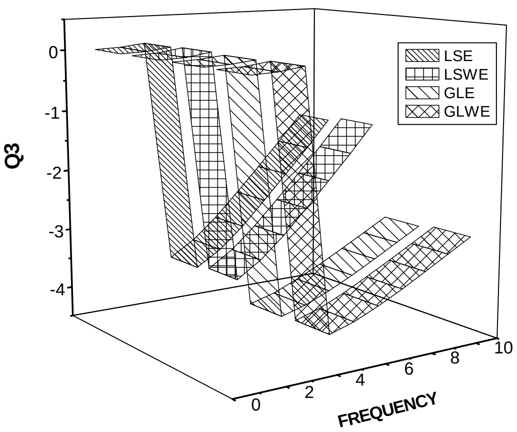

Fig. 8. Variation of attenuation coefficient Q_3 with frequency ω.

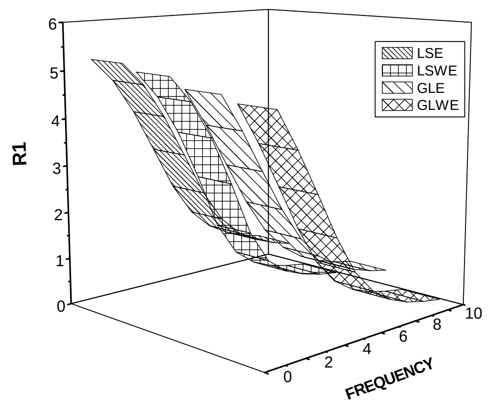

Fig. 9. Variation of specific loss R_1 with frequency ω.

8.4. Penetration depth. It is evident from Fig. 12 that the values of penetration depth S_1 increase rapidly for LSWE and GLWE within the range $1 \le \omega \le 10$. Figure 13 shows that the values of S_2 remain oscillatory for smaller values of ω , but a respectable electro effect can be noticed for LSE and GLE within the range $8 \le \omega \le 10$.

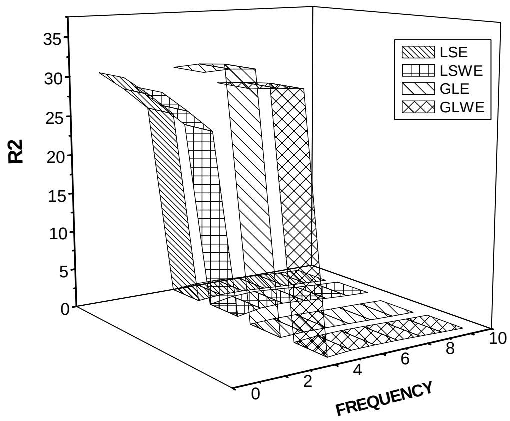

Fig. 10. Variation of specific loss R_2 with frequency ω.

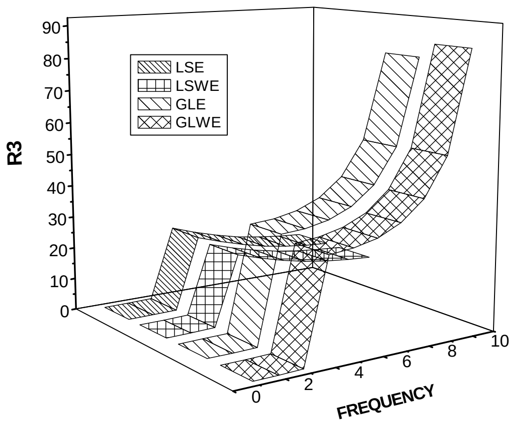

Fig. 11. Variation of specific loss R_3 with frequency ω.

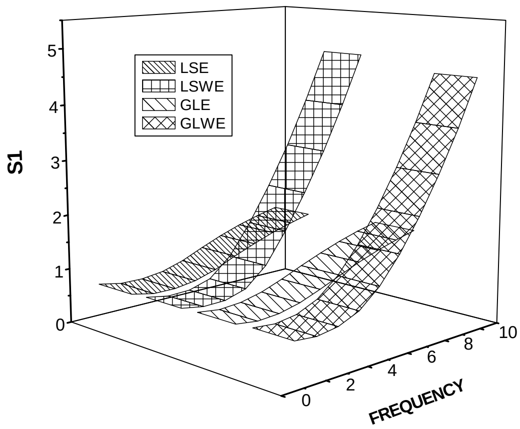

Fig. 12. Variation of penetration depth S_1 with frequency ω.

In both Fig. 12 and Fig. 13, due to presence of electro effect, a significant difference in the trend of variation and behavior of penitration depth for LSWE and GLWE can be noticed in comparison to LSE and GLE respectively. The value of penetration depth S_3 decrease for smaller values of ω, but for higher values of ω, the values of S_3 becomes stationary as shown in Fig. 14.

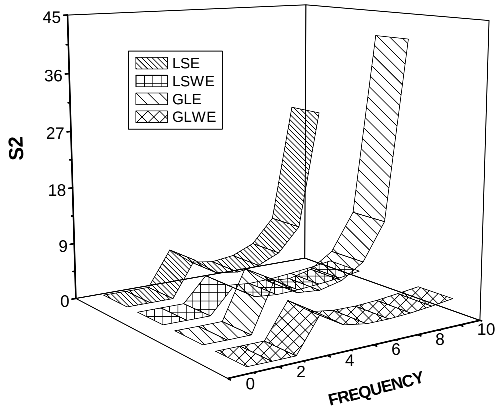

Fig. 13. Variation of penetration depth S_2 with frequency ω.

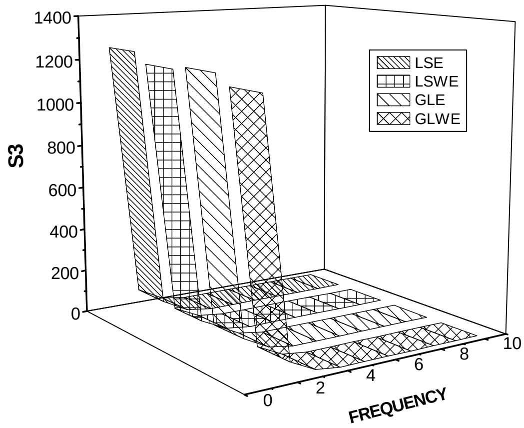

Fig. 14. Variation of penetration depth S_3 with frequency ω.

9. Conclusions

The propagation of plane waves and fundamental solution in a homogeneous, isotropic electro-microstretch generalized thermoelastic solid medium has been investigated. For two dimensional model, there exist three coupled waves, namely longitudinal displacement wave (LD-wave), thermal wave (T-wave), longitudinal microstretch wave (LM-wave) and two coupled transverse displacement and transverse microrotational (CD I, CD II) waves. The phase velocity for T, LM, CD-I and CD-II waves remains more for GLE and GLWE in comparison to LSE and LSWE theories. The magnitude of R_1, R_2 and S_3 decreases for all the theories and becomes constant. The investigation of thermal and electro effects on micropolar elastic wave propagation plays an important role in understanding many seismological processes.

The phase velocities, attenuation coefficients, specific loss and penetration depth of longitudinal waves are computed and presented graphically with respect to frequency. The fundamental solution $G(x)$ of the system of equations (4.7) makes it possible to investigate three- dimensional boundary value problems of generalized theory of electro-microstretch thermoelastic solids by potential method [30].

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