Materials Physics and Mechanics 16 (2013) 66-81 Received: December 9, 2012

PROPAGATION OF WAVES IN MICROPOLAR THERMOELASTIC SOLID WITH TWO TEMPERATURES BORDERED WITH LAYERS OR HALF-SPACES OF INVISCID LIQUID

Kunal Sharma[1*], Saurav Sharma[2], Raj Rani Bhargava[3***]**

[1]Department of Mechanical Engineering, Ecole polytechnique fédérale de Lausanne (EPFL), Lausanne, Switzerland

[2]Department of Instrumentation, Kurukshetra University, Kurukshetra, India

[3]Department of Mathematics IIT Roorkee, Roorkee 247667, India

*e-mail: kunal.sharma@epfl.ch

**e-mail: sauravkuk@gmail.com

***e-mail: rajrbfma@iitr.ernet.in

Abstract. The present study is concerned with the propagation of Lamb waves in a homogeneous isotropic thermoelastic micropolar solid with two temperatures bordered with layers or half-spaces of inviscid liquid subjected to stress free boundary conditions. The coupled thermoelasticity theory has been used to investigate the problem. The secular equations for symmetric and skew-symmetric leaky and nonleaky Lamb wave modes of propagation are derived. The phase velocity and attenuation coefficient are computed numerically and depicted graphically. The amplitudes of stress, microrotation vector and temperature distribution for the symmetric and skew-symmetric wave modes are computed numerically and presented graphically. Results of some earlier workers have been deduced as particular cases.

1. Introduction

The exact nature of layers beneath the earth's surface are unknown. Therefore, one has to consider various appropriate models for the purpose of theoretical investigation. Modern engineering structures are often made up of materials possessing an internal structure. Polycrystalline materials, materials with fibrous or coarse grain structure come in this category. Classical elasticity is inadequate to represent the behaviour of such materials. The analysis of such materials requires incorporating the theory of oriented media. For this reason, micropolar theories were developed by Eringen [1-3] for elastic solids, fluids and further for non-local polar fields and are now universally accepted. A micropolar continuum is a collection of interconnected particles in the form of small rigid bodies undergoing both translational and rotational motions.

The linear theory of micropolar thermoelasticity was developed by extending the theory of micropolar continua to include thermal effects by Eringen [4] and Nowacki [5]. Dost and Taborrok [6] presented the generalized thermoelasticity by using Green and Lindsay theory [7].

The main difference of thermoelasticity with two temperatures with respect to the classical one is the thermal dependence. Chen et al. [8, 9] have formulated a theory of heat

$$(\alpha+\beta+\gamma)\nabla\left(\nabla.\vec{\phi}\right)-\gamma\nabla\times\left(\nabla\times\vec{\phi}\right)+K\nabla\times\vec{u}-2K\vec{\phi}=\rho j\frac{\partial^2\vec{\phi}}{\partial t^2}, \tag{2}$$

$$K^*\nabla^2\Phi=\rho c^*\left(\frac{\partial}{\partial t}\right)(1-a\nabla^2)\Phi+\nu\Phi_0\left(\frac{\partial}{\partial t}\right)\left(\nabla.\vec{u}\right), \tag{3}$$

and the constitutive relations are

$$t_{ij}=\lambda u_{r,r}\delta_{ij}+\mu\left(u_{i,j}+u_{j,i}\right)+K\left(u_{j,i}-\varepsilon_{ijr}\phi_r\right)-\nu T\delta_{ij}, \tag{4}$$

$$m_{ij}=\alpha\phi_{r,r}\delta_{ij}+\beta\phi_{i,j}+\gamma\phi_{j,i}, \qquad i,j,r=1,2,3 \tag{5}$$

where ∇^2 is the Laplacian operator; λ and μ are Lame's constants; K, α, β and γ are micropolar constants; t_{ij} are the components of the stress tensor, and m_{ij} are the components of couple stress tensor; $\vec{u}$ and $\vec{\phi}$ are the displacement and microrotation vectors; ρ is the density; j is the microinertia; K^* is the thermal conductivity; c^* is the specific heat at constant strain; T is the temperature change; $\nu=(3\lambda+2\mu+K)\alpha_T$, where α_T is the coefficient of linear thermal expansio; δ_{ij} is the Kronecker delta; ε_{ijr} is the alternating symbol; T and Φ are connected by the relation $T=(1-a\nabla^2)\Phi$.

For the liquid half-space, the equation of motion and constitutive relations are given by

$$\lambda_L\nabla(\nabla.\overrightarrow{u_L})=\rho_L\frac{\partial^2\overrightarrow{u_L}}{\partial t^2}, \tag{6}$$

$$(t_{ij})_L=\lambda_L(u_{r,r})_L\delta_{ij}. \tag{7}$$

3. Formulation of the problem

Consider an infinite homogeneous isotropic, thermally conducting micropolar thermoelastic plate of thickness $2d$ initially undisturbed and at uniform temperature T_0. The plate is bordered with infinitely large homogeneous inviscid liquid half-spaces or layers of thickness h on both sides as illustrated in Figs. 1(a) and 1(b). We take origin of the co-ordinate system (x_1,x_2,x_3) on the middle surface of the plate and x_1-axis is taken normal to the solid plate.

For two dimensional problem, we take

$$\vec{u}=\left(u_1(x_1,x_3),0,u_3(x_1,x_3)\right), \qquad \vec{\phi}=\left(0,\phi_2(x_1,x_3),0\right). \tag{8}$$

For convenience, the following non dimensional quantities are introduced

$$x_1'=\frac{\omega^*x_1}{c_1}, \quad x_3'=\frac{\omega^*x_3}{c_1}, \quad u_1'=\frac{\rho\omega^*c_1}{\nu T_0}u_1, \quad u_3'=\frac{\rho\omega^*c_1}{\nu T_0}u_3,$$

$$\phi_2'=\frac{\rho c_1^{\,2}}{\nu T_0}\phi_2, \quad t'=\omega^*t, \quad T'=\frac{T}{T_0}, \quad \Phi'=\frac{\Phi}{\Phi_0}, \quad t_{ij}'=\frac{1}{\nu T_0}t_{ij}, \quad m_{ij}'=\frac{\omega^*}{c_1\nu T_0}m_{ij},$$

$$u_L'=\frac{\rho\omega^*c_1}{\nu T_0}u_L, \quad w_L'=\frac{\rho\omega^*c_1}{\nu T_0}w_L, \quad c_L^2=\frac{\lambda_L}{\rho_L}, \quad h'=\frac{c_1h}{\omega^*}, \quad d'=\frac{\omega^*d}{c_1}, \quad a'=\frac{\omega^{*2}}{c_1^2}a, \tag{9}$$

$$\nabla^2\phi_{L_i} - \frac{1}{\delta_L^2}\frac{\partial^2\phi_{L_i}}{\partial t^2} = 0, \quad i = 1,2\,, \tag{16}$$

where $a_1 = \dfrac{K}{\mu + K}$, $a_2 = \dfrac{\rho\, c_1^2}{\mu + K}$, $a_3 = \dfrac{Kc_1^2}{\gamma\omega^{*2}}$, $a_4 = 2\, a_3$, $a_5 = \dfrac{\rho\hat{j}c_1^2}{\gamma}$,

$a_6 = \dfrac{\rho\, c^* c_1^2}{K^*\omega^*}$, $a_7 = \dfrac{\nu^2 T_0}{\rho \mathrm{K}^*\omega^*}$, $\nabla^2 = \dfrac{\partial^2}{\partial {x_1}^2} + \dfrac{\partial^2}{\partial {x_3}^2}$, $\delta_L^2 = \dfrac{c_L^2}{c_1^2}$.

The shear motion is not supported by inviscid fluid, therefore shear modulus of liquid vanishes and hence ψ_{L_i}, $i = 1,2$ vanish. In case of inviscid liquid, the potential function layer satisfy the equation (16).

Consider the propagation of plane waves in the x_1x_3-plane with a wavefront parallel to the x_2-axis, therefore, ϕ, ψ, ϕ_2, Φ, ϕ_{L_1} and ϕ_{L_2} are independent of x_2-coordinates.
We assume the solutions of Eqs. (12)-(16) of the form

$$(\phi,\, \psi, \phi_2, \Phi, \phi_{L_1}, \phi_{L_2}) = \left[f_1(x_3), f_2(x_3), f_3(x_3), f_4(x_3), f_5(x_3), f_6(x_3)\right]e^{i\xi(x_1 - ct)}, \tag{17}$$

where c=$\dfrac{\omega}{\xi}$ is the non-dimensional phase velocity, ω is the frequency and ξ is the wave number.

Using Eq. (17) in Eqs. (12)-(16), we obtain

$$(\nabla^{*2} + \xi^2 c^2) f_1(x_3) = \left[1 - a\nabla^{*2}\right] f_4(x_3), \tag{18}$$

$$(\nabla^{*2} + a_6 i\xi c(1 - a\nabla^{*2})) f_4(x_3) = a_7 i\xi c\nabla^{*2} f_1(x_3), \tag{19}$$

$$(\nabla^{*2} + a_2\xi^2 c^2) f_2(x_3) = -a_1 f_3(x_3), \tag{20}$$

$$(d^2 / d{x_3}^2 - {\gamma_L}^2) f_k(x_3) = 0, \quad (k = 5, 6)\,, \tag{21}$$

$$(\nabla^{*2} + a_5\xi^2 c^2) f_3(x_3) = a_3\nabla^{*2} f_2(x_3). \tag{22}$$

Eliminating $f_4(x_3)$ from Eqs. (18) and (19) and eliminating $f_3(x_3)$ from Eqs. (20) and (22) yield

$$(\nabla^{*4} + A\nabla^{*2} + B) f_1(x_3) = 0, \tag{23}$$

$$(\nabla^{*4} + C\nabla^{*2} + D) f_2(x_3) = 0, \tag{24}$$

where $\nabla^{*2} = d^2 / d{x_3}^2 - \xi^2$ and A, B, C and D are given by

$$A = \left(aa_6 - \frac{1}{\xi^2 c^2} - \frac{a_6 i}{\xi^3 c^3} - p_0 a_7 i\right) / \left(aa_6 - \frac{1}{\xi^2 c^2} - p_0 aa_7 i\right), \quad B = \left(-i\xi ca_6\right) / \left(aa_6 - \frac{1}{\xi^2 c^2} - p_0 aa_7 i\right),$$

(iii) The tangential component of the couple stress tensor should be zero.

$$(m_{32})_s = 0. \tag{33}$$

(iv) The normal velocity component of the solid should be equal to that of the liquid.

$$(\dot{u}_3)_s = (\dot{w})_L. \tag{34}$$

(v) The thermal boundary conditions is given by

$$\frac{\partial T}{\partial x_3} + HT = 0, \tag{35}$$

where H is the surface heat transfer coefficient. Here $H \to 0$ corresponds to thermal insulated boundaries and $H \to \infty$ refers to isothermal one.

4.1. Leaky Lamb waves. The solutions for solid media of finite thickness $2d$ sandwiched between two liquid half-spaces is given by equations (25)-(28) and

$$\phi_{L_1} = E_5 e^{\gamma_L (x_3+d)} e^{i\xi(x_1-ct)},\ -\infty < x_3 < -d, \tag{36}$$

$$\phi_{L_2} = F_6 e^{-\gamma_L (x_3-d)} e^{i\xi(x_1-ct)},\ d < x_3 < \infty. \tag{37}$$

4.2. Nonleaky Lamb waves. The corresponding solutions for a solid media of finite thickness $2d$ sandwiched between two finite liquid layers of thickness h is given by equations (25)-(28) and

$$\phi_{L_1} = E_5 \sinh \gamma_L [x_3 + (d+h)] e^{i\xi(x_1-ct)},\ -(d+h) < x_3 < -d, \tag{38}$$

$$\phi_{L_2} = F_6 \sinh \gamma_L [x_3 - (d+h)] e^{i\xi(x_1-ct)},\ d < x_3 < (d+h). \tag{39}$$

Nonleaky and leaky Lamb waves are distinguished by selecting the functions ϕ_{L_1} and ϕ_{L_2} in such a way that the acoustical pressure is zero at $x_3 = \mp(d+h)$. This shows that ϕ_{L_1} and ϕ_{L_2} are solutions of standing wave and travelling wave for nonleaky Lamb waves and leaky Lamb waves respectively.

5. Derivation of the dispersion equations

We apply the already shown formal solutions in this section to study the specific situations with inviscid fluid.

5.1. Leaky Lamb waves. Consider an isotropic thermoelastic micropolar plate with two temperatures completely immersed in the inviscid liquid as shown in Fig. 1(a). The thickness of the plate is $2d$ and thus the lower and upper portions of the fluid extend from $x_3 = d$ to ∞ and $x_3 = -d$ to $-\infty$ respectively. In this case, the partial waves are in both the plate and the fluid. The appropriate formal solutions for the plate and fluid are those given by equations (25)-(28), (36) and (37). By applying the boundary conditions (31)-(35) at $x_3 = \pm d$ and subsequently requiring nontrivial values of the partial wave amplitudes E_k and F_k, (k=1, 2, 3, 4), E_5, F_6 and $\gamma_L \neq 0$, we arrive at the characteristic dispersion equations as

$$m_i = [d_1 l_i + d_2 n_i^2 + b_i h_i],$$

$$m_3 = (2d_4 + d_5)i\xi, \quad m_k = (d_4 + d_5)n_j^2 - d_4\xi^2 - d_5 h_j, \quad i = 1,2, \quad j = 3,4, \quad k = 5,6\,,$$

$$S = \frac{\rho_L}{\rho}\xi^2 c^2, \quad R = \iota\xi c, \quad Q = \iota\xi d_2, \quad l_i = \xi^2 + n_i^2, G = -i\xi c\gamma_L, \; b_i = p_0(1 + a\xi^2 + an_i^2),$$

$$s_i = \sin m_i d, \quad s_j = \sin m_j d, \quad c_i = \cos m_i d, \quad c_j = \cos m_j d, \quad T_5 = \tanh \gamma_L h,$$

$$d_1 = \frac{\lambda}{\rho c_1^2}, \quad d_2 = \frac{(2\mu + \kappa)}{\rho c_1^2}, \quad d_4 = \frac{2\mu}{\rho c_1^2}, \quad d_5 = \frac{d_2}{2},$$

$$T_i = \tan m_i d, \qquad i = 1,2,3,4\,.$$

Here the superscript +1 refers to skew-symmetric and -1 refers to symmetric modes.

Equations (40) and (43) are the dispersion relations involving wave number and phase velocity of various modes of propagation in a micropolar thermoelastic plate with two temperatures bordered with layers of inviscid liquid or half-spaces on both sides.

6. Special cases

If the liquid layers or half-spaces on both sides are removed, then we are left with the problem of wave propagation in micropolar thermoelastic solid with two temperatures. For this, we take $\rho_L = 0$ in equations (40) and (42), the secular equations for stress free thermally insulated boundaries ($H \to 0$) for the said case reduce to

$$(T_1T_3)^{\pm} AT1 + (T_1T_4)^{\pm} AT2 + (T_2T_3)^{\pm} AT3 + (T_2T_4)^{\pm} AT4 + (T_3T_4)^{\pm} AT5 = 0\,.$$

Subcase (i):

In this case, if $a = 0$, we obtain the secular equations in micropolar generalized thermoelastic plate.

7. Amplitudes of dilatation, microrotation and temperature distribution

In this section the amplitudes of dilatation, microrotation and temperature distribution for symmetric and skew-symmetric modes of waves have been computed for micropolar thermoelastic plate. Using Eqs. (18)-(25) and (28)-(35), we obtain

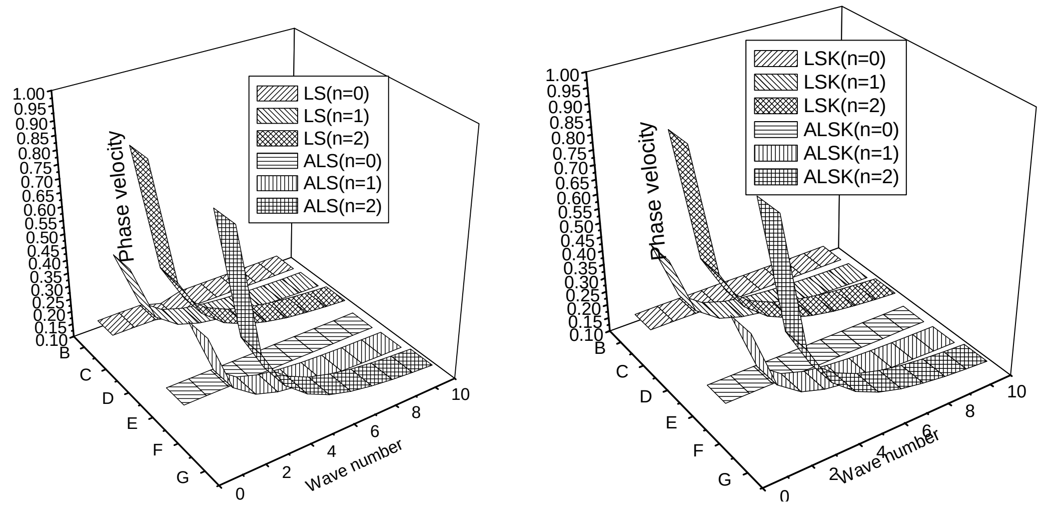
$$(e)_{sy} = [-M_1 \cos n_1 x_3 + \frac{M_2 L s_1}{s_2} \cos n_2 x_3] A_1 e^{i\xi(x_1 - ct)}, \tag{44}$$

$$(e)_{asy} = [-M_1 \sin n_1 x_3 + \frac{M_2 L c_1}{c_2} \sin n_2 x_3] B_1 e^{i\xi(x_1 - ct)}, \tag{45}$$

$$(\phi_2)_{sy} = [h_3 \sin n_3 x_1 - \frac{h_3 n_3 c_3}{n_4 c_4} \sin n_4 x_3] B_3 e^{i\xi(x_1 - ct)}, \tag{46}$$

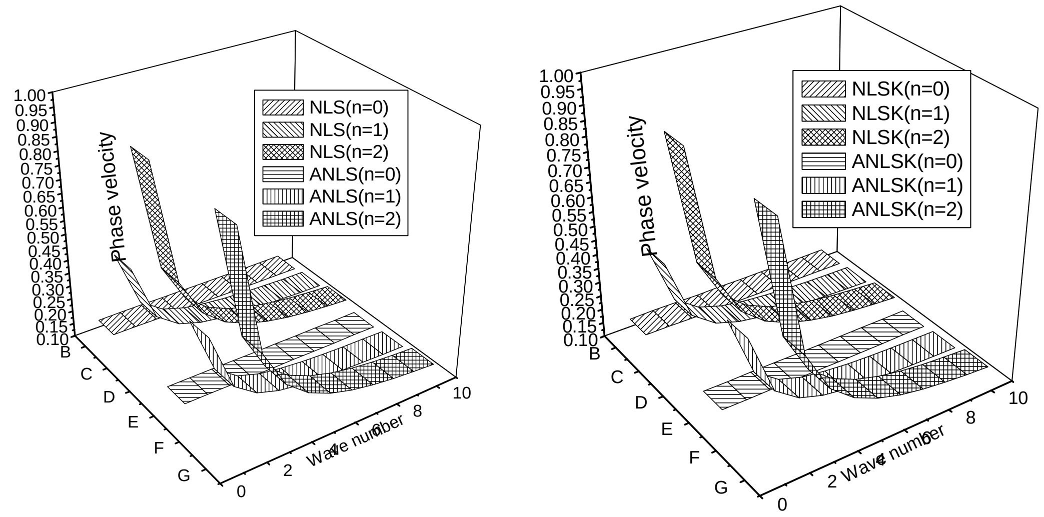
$$(\phi_2)_{asy} = [h_3 \cos n_3 x_3 - \frac{h_3 n_3 s_3}{n_4 s_4} \cos n_4 x_3] A_3 e^{i\xi(x_1 - ct)}, \tag{47}$$

velocities for ALS for wave number $\xi d = 2, 4, 5, 6, 7, 8, 10$ and for (n=1) symmetric nonleaky Lamb wave mode of propagation, the velocities for NLS and ANLS coincide. It is noticed that for (n=2) symmetric nonleaky Lamb wave modes of propagation, the phase velocitiy for NLS remain more than in case of ANLS for wave number $\xi d = 3, 5$ and the behavior is reversed for $\xi d = 2$ and in the remaining range, the phase velocities coincide for NLS and ANLS. For symmetric leaky Lamb wave mode of propagation (n=2), the phase velocities for LS remain more than the velocities for ALS for wave number $\xi d = 1, 2$ and then coincide.



(a) (b)

Fig. 2. Variation of phase velocity for symmetric (a) and skew-symmetric (b) leaky Lamb waves.

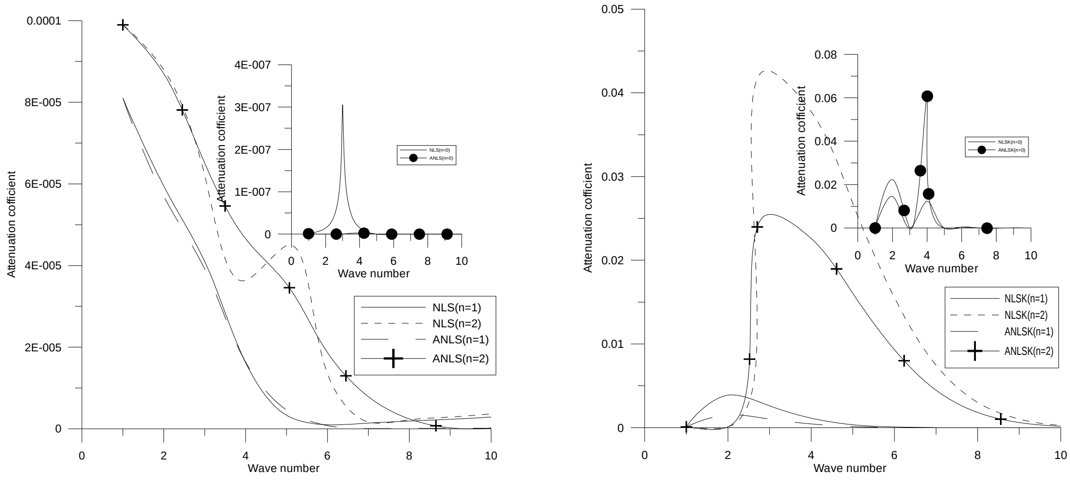


(a) (b)

Fig. 3. Variation of phase velocity for symmetric (a) and skew-symmetric (b) nonleaky Lamb waves.

It is observed from Fig. 2(b) that the phase velocities for lowest skew-symmetric leaky Lamb wave mode of propagation for LSK and ALSK coincide.There is minute difference in

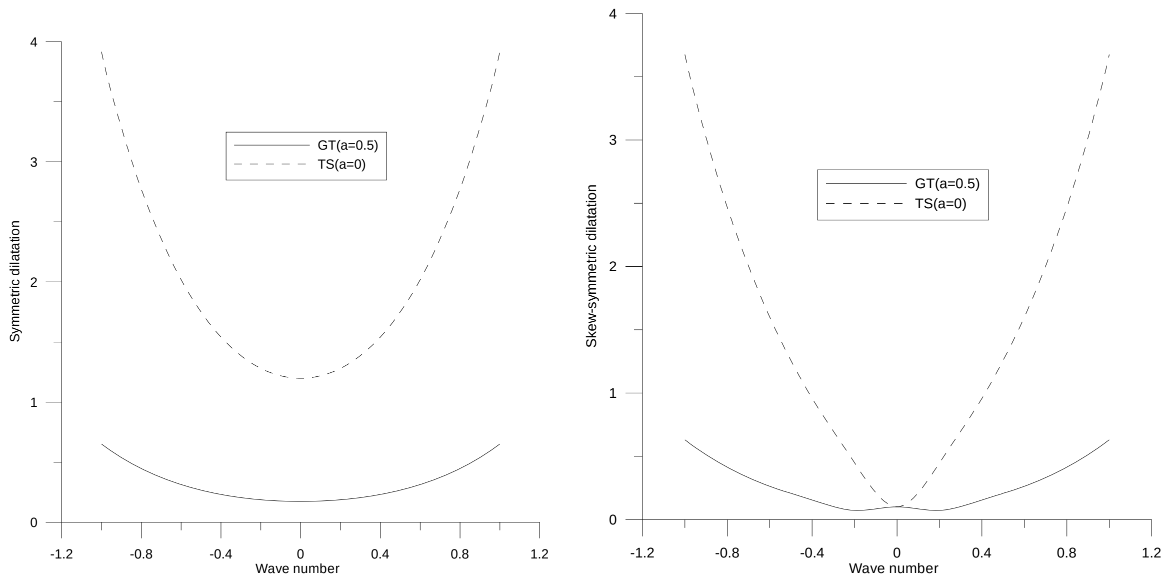
value of 0.0145 at $\xi d = 2$ and ANLSK attain maximum valu at $\xi d = 4$. It is observed that for (n=1), the magnitude for NLSK remain more than the values for ANLSK in the region $2 \le \xi d \le 8$ and the behavior is reversed in the remaining region. For (n=2) mode, NLSK attain maximum value 0.0428 and ANLSK attain maximum value of 0.0250 at $\xi d = 3$.



(a) (b)

Fig. 5. Variation of attenuation coefficient for symmetric (a) and skew-symmetric (b) nonleaky Lamb waves.

8.3. Amplitudes. TS represents the amplitude for micropolar thermoelastic solid and GT represents the amplitude for micropolar thermoelastic solid with two temperatures in Figs. 6 to 8.



(a) (b)

Fig. 6. Variation of symmetric (a) and skew-symmetric (b) dilatation.

Variations of symmetric and skew-symmetric amplitudes of dilatation for LS theory for stress free thermally insulated boundary are depicted in Figs. 6(a) to 6(b). The dilatation is

and then coincide with increase in wave number. Also the phase velocities for higher symmetric and skew-symmetric mode attain peak value at vanishing wave number and as wave number increases the phase velocities decrease sharply. The values of attenuation cofficient of (n=2) skew-symmetric leaky and nonleaky Lamb waves mode for TWST remain higher than that in case of TS. It is also noticed that the values of attenuation cofficient for lowest symmetric and skew-symmetric mode for leaky and non leaky Lamb waves are very small as compared to the values for highest mode. The values of symmetric and skew-symmetric dilatation in case of TS are greater in comparison to that of GT and the values of symmetric temperature in case of TS are higher than that in case of GT, while the values of skew-symmetric temperature in case of TS are higher.

References

[1] A.C. Eringen // *J. Math. Mech.* **15** (1966) 909.

[2] A.C. Eringen // J. *Math. Mech.* **16** (1966) 1.

[3] A.C. Eringen, In: *Continuum Physics*, ed. by A.C. Eringen (Academic Press, New York, 1976), Vol. IV, p. 205.

[4] A.C. Eringen, *Foundations of micropolar thermoelasticity*, International Centre for Mechanical Science, Udline Course and Lectures 23 (Springer-Verlag, Berlin, 1970).

[5] W. Nowacki, *Theory of Asymmetric Elasticity* (Pergamon, Oxford, 1986)

[6] S. Dost, B. Taborrok // *Int. J. Eng. Sci.* **16** (1978) 173.

[7] A.E. Green, K.A. Lindsay // *J. of Elasticity* **2** (1972) 1.

[8] P.J. Chen, M.E. Gurtin, W.O. Williams // *Zeitschrift für angewandte Mathematik und Physik* ***19*** (1968) 960.

[9] P.J. Chen, M.E. Gurtin, W.O. Williams // *Zeitschrift für angewandte Mathematik und Physik* **20** (1969) 107.

[10] M. Boley // *J. Appl. Phys.* **27** (1956) 240.

[11] W.E. Warren, P.J. Chen // *Acta Mechanica* **16** (1973) 21.

[12] H.M. Youssef // *IMA Journal of Applied Mathematics* **71** (2006) 383.

[13] P. Puri, P. Jordan // *Int. J. of Eng. Sci.* **44** (2006) 1113.

[14] H.M. Youssef, E.A. Al-Lehaibi // *Int. J. of Solid and Struct*. **44** (2007) 1550.

[15] H.M. Youssef, H.A. Al-Harby // *Archive Applied Mechanics* **77** (2007) 675.

[16] A. Magana, R. Quintanilla // Math. and Mech. of Solids, Online (2008).

[17] S. Mukhopadhyay, R. Kumar // *J. of Thermal Stresses* **32** (2009) 341.

[18] K. Roushan, M. Santwana // *Int. J. Eng. Sci.* 48 (2010) 128-139.

[19] S. Kaushal, N. Sharma, R. Kumar // *Int. J. of Appl. Mech. and Eng.* **15** (2010) 1111.

[20] S. Kaushal, R. Kumar, A. Miglani // *Mathematical Sciences* **5** (2011) 125.

[21] M.A. Ezzat, E.S. Aiwad // *J. of Thermal Stresses* **33** (2010) 226.

[22] M.A. Ezzat, F. Hamza, E. Awad // *Acta Mechanica Solida Sinica* **23** (2010) 200.

[23] A.H. Nayfeh, *Wave Propagation in Layered Anisotropic Media: With Application to Composites* (Elsevier Science, North-Holland, 1995)

[24] Q. Qi // *Journal of Acoustical Society of America* **95** (1994) 3222.

[25] J. Wu, Z. Zhu // *Journal of Acoustical Society of America* **97** (1995) 3191.

[26] Z. Zhu, J. Wu // *Journal of Acoustical Society of America* **98** (1995)1059.

[27] A.H. Nayfeh, P.B. Nagy // *Journal of Acoustical Society of America* **101** (1997) 2649.

[28] W. Nowacki, W.K. Nowacki // *Bulletin de L'Academie Polonaise Des Science: Série des sciences techniques* **17** (1969) 45.

[29] R. Kumar, M.L. Gogna // *Int. J. of Eng. Sci.* **27** (1988) 89.

[30] S.K. Tomar // *Proceedings of the National Academy of Sciences, India Section A: Physical Sciences* **72** IV (2002) 339.

[31] S.K. Tomar // *J. of Vibration and Control* **11** (2005) 849.

[32] R. Kumar, G. Pratap // *Appl. Math. and Mech.* **27** (2006) 1049.
[33] R. Kumar, G. Pratap // *Int. J. of Appl. Mech. and Eng.* **12** (2007) 655.
[34] R. Kumar, G. Pratap // *Buletinul Institutului Politehnic din Iaşi* **3-4** (2007) 53.
[35] R. Kumar, G. Pratap // *Int. J. of Appl. Math. and Mech*. **4** (2008) 19.
[36] R. Kumar, G. Pratap // *Appl. Math. and Inf. Sci.* **5** (2009) 39.
[37] R. Kumar, G. Pratap // *Appl. Math. and Inf. Sci.* **4** (2010) 107.
[38] J.N. Sharma, S. Kumar, Y.D. Sharma // *J. of Thermal Stresses* **31** (2008) 18.
[39] J.N. Sharma, S. Kumar // *Meccanica* **44** (2009) 305.