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FUNDAMENTAL SOLUTION FOR TWO-DIMENSIONAL PROBLEM IN ORTHOTROPIC PIEZOTHERMOELASTIC DIFFUSION MEDIA

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Abstract. The present investigation deals with the study of two-dimensional fundamental solution in orthotropic piezothermoelastic diffusion media. By virtue of the two-dimensional general solution of orthotropic piezothermodiffusion elastic media, the fundamental solution for a point heat source and chemical potential source on the surface of a semi-infinite orthotropic piezothermoelastic diffusion plane is constructed by five newly introduced harmonic functions. The components of displacement, stress, electric displacement, electric potential, temperature change and chemical potential are expressed in terms of elementary functions. The components of displacement, electric potential, temperature change and chemical potential are computed numerically and depicted graphically. From the present investigation, a special case of interest is also deduced to depict the effect of diffusion.

1. Introduction

Fundamental solutions or Green's functions play an important role in the solution of numerous problems in the mechanics and physics of solids. They are a basic building of the block of many further works. For example, fundamental solution can be used to construct many analytical solutions of practical problems, when boundary conditions are imposed. They are also very important in the boundary element method as well as the study of cracks, defects and inclusions.

Many studies have been directed towards obtaining full space Green's function for elastic solid in isotropic and anisotropic media due to Lord Kelvin [1], Freedholm [2], Synge [3], Pan and Chou [4]. Fourier transform method has been applied to investigate the Green's functions by Lifshitz and Rezentsveig [5] and Lejcek [6]. For anisotropic piezoelectric media, one can refer to important work of Deeg [7], Wang [8], Benveniste [9], and Chen and Lin [10]. Lee and Jiang [11] studied the boundary integral formulation and two-dimensional fundamental solution for piezoelectric media. Wang and Zheng [12] investigated the general solution for three-dimensional problem in piezoelectric media. Ding et al. [13] investigated the Fundamental solution for piezoelectric media. Ding et al. [14] studied the Fundamental solution for plane problem of piezoelectric materials.

The thermal effect is not considered in the above works. Rao and Sunar [15] pointed out the temperature variation in the piezoelectric media. Qin [16] derived the two-dimensional Green's functions of anisotropic pyroelectric media with holes of various shapes. Chen et al. [17] derived the general solution for transversely isotropic piezothermoelatic media. In this general solution all components of the coupled field are expressed by four harmonic functions. On the basis of this general solution Chen et al. [18] obtained Green's function of transversely isotropic pyroelectric media with a penny shaped. Hou et al. [19]

constructed Green's function for a point heat source on the surface of a semi-infinite transversely isotropic pyroelectric media. Hou [20] investigated the two-dimensional fundamental solution for orthotropic pyroelectric media.

Diffusion can be defined as the movement of particles from an area of high concentration to an area of lower concentration until equilibrium is reached. It occurs as a result of second law of thermodynamic which states that the entropy or disorder of any system must always increase with time. Diffusion is important in many life processes. Now days, there is a great deal of interest in the study of this phenomena, due to its many application in geophysics and industrial applications. Until recently, thermodiffusion in solids, especially in metals, was considered as a quantity that is independent of body deformation. Practice however indicates that the process of thermodiffusion could have a very considerable influence on the deformation of the body. Thermodiffusion in elastic solid is due to the coupling of temperature, mass diffusion and strain in addition to the exchange of heat and mass with the environment.

Nowacki [21-24] developed the theory of thermoelastic diffusion by using coupled thermoelastic model. This implies infinite speed of propagation of thermoelastic waves. Kumar and Chawla [25] discussed the Plane wave propagation in anisotropic three-phase-lag and two-phase-lag model. Kuang [26] discussed the variational principles for generalized thermodiffusion theory in pyroelectricity. Kumar and Chawla [27] obtained the two-dimensional Fundamental solution in orthotropic thermoelastic diffusion media. Kumar and Chawla [28] derived the Green's functions for two-dimensional problem in orthotropic thermoelastic diffusion media. Also Kumar and Chawla [29] obtained the three-dimensional fundamental solution in transversely isotropic media. However the important fundamental solution in piezothermoelastic diffusion has not been discussed so far.

The fundamental solution for two-dimensional fundamental solution in orthotropic piezothermoelastic diffusion media is investigated in this paper. With this objective, the general solution in orthotropic piezothermoelastic media is derived at first. On the basis of general solution, the two-dimensional fundamental solution for a point heat source and chemical potential source on the surface of a semi-infinite orthotropic piezothermoelastic diffusion medium is presented by five newly introduced harmonic functions. The components of displacement, electric potential, temperature change and chemical potential are computed numerically and presented graphically.

2. Basic equations

The basic governing equations of orthotropic piezothermoelastic diffusion media can be found in Ref. [26]. If all the components are independent of coordinate y, so called the plane problem, the constitutive equations in two-dimensional Cartesian coordinate (x, z) can be expressed as

$$\sigma_{xx} = c_{11}\frac{\partial u}{\partial x} + c_{13}\frac{\partial w}{\partial z} + e_{31}\frac{\partial \Phi}{\partial z} - \beta_1 T - b_1\mu, \tag{1}$$

$$\sigma_{zz} = c_{13}\frac{\partial u}{\partial x} + c_{33}\frac{\partial w}{\partial z} + e_{33}\frac{\partial \Phi}{\partial z} - \beta_3 T - b_3\mu, \tag{2}$$

$$\sigma_{zx} = c_{55}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + e_{15}\frac{\partial \Phi}{\partial x}, \tag{3}$$

$$D_x = e_{15}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) - \varepsilon_{11}\frac{\partial \Phi}{\partial x}, \tag{4}$$

$$D_z = e_{31}\frac{\partial u}{\partial x} + e_{33}\frac{\partial w}{\partial z} - \varepsilon_{33}\frac{\partial \Phi}{\partial z} + p_3 T, \tag{5}$$

where u and w are components of the mechanical displacement in x and z directions, respectively; σ_{ij} and D_i are the components of stress and electric displacement, respectively; β_i and b_i are material constants. Φ and T are electric potential and temperature increment, respectively; $c_{ij}, e_{ij}, \varepsilon_{ij}$, and p_3 are elastic piezoelectric, dielectric, thermal modules, diffusion modules and pyroelectric contants, respectively.

The mechanical, electric, heat equilibrium and mass diffusions equations for static problem, in the absence of body forces, free charges, heat sources and mass diffusive sources are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial z} = 0, \qquad \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = 0, \tag{6}$$

$$\frac{\partial D_x}{\partial x} + \frac{\partial \sigma_z}{\partial z} = 0, \tag{7}$$

$$\left(\lambda_1 \frac{\partial^2}{\partial^2 x} + \lambda_3 \frac{\partial^2}{\partial^2 z}\right) T = 0, \tag{8}$$

$$\left(D_1 \frac{\partial^2}{\partial^2 x} + D_3 \frac{\partial^2}{\partial^2 z}\right) \mu = 0. \tag{9}$$

We define the dimensionless quantities:

$$x' = \frac{\omega_1^* x_i}{v_1}, \quad z' = \frac{\omega_1^* z}{v_1}, \quad u' = \frac{\omega_1^* u}{v_1}, \quad w' = \frac{\omega_1^* w}{v_1}, \quad \mathrm{T}' = \frac{\mathrm{T}}{\mathrm{T}_0}, \quad \Phi' = \frac{\omega^* e_{33}\Phi}{v_1 \beta_1 T_0}, \quad \mu' = \frac{\mu}{v_1^2},$$
$$\sigma_{ij}' = \frac{\sigma_{ij}}{\beta_1 \mathrm{T}_0}, \quad D_i' = \frac{D_i}{\sqrt{\beta_1 T_0}}, \quad H' = \frac{v_1}{T_0 \lambda_1 \omega_1^*} H, \quad P' = \frac{P}{D_1 v_1 \omega_1^*} H,$$

where

$$v_1^2 = \frac{\beta_1 T_0}{b_1}, \quad \omega_1^* = \frac{\beta_1 c_{11}}{b_1 \lambda_1}. \tag{10}$$

Substituting equations (1)-(5) into equations (6)-(7) and applying the dimensionless quantities defined by Eq. (10) on resulting equations, after suppressing the primes, we obtain

$$\left(\frac{\partial^2}{\partial x^2} + \delta_1 \frac{\partial^2}{\partial z^2}\right) u + \left(\delta_2 \frac{\partial^2}{\partial x \partial z}\right) w - e_1 \bar{\varepsilon}_p \frac{\partial^2 \Phi}{\partial x \partial z} - r_1\left(\frac{\partial}{\partial x}\right) T - q_1\left(\frac{\partial}{\partial x}\right)\mu = 0, \tag{11}$$

$$\left(\delta_2 \frac{\partial^2}{\partial x \partial z}\right) u + \left(\delta_1 \frac{\partial^2}{\partial x^2} + \delta_3 \frac{\partial^2}{\partial z^2}\right) w - \bar{\varepsilon}_p \left(e_2 \frac{\partial^2}{\partial x^2} + \delta_3 \frac{\partial^2}{\partial z^2}\right)\Phi - r_3\left(\frac{\partial}{\partial z}\right) T - q_3\left(\frac{\partial}{\partial z}\right)\mu = 0, \tag{12}$$

$$\left(e_1\frac{\partial^2}{\partial x\partial z}\right)u+\left(e_2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)w-\bar{\varepsilon}_q\left(\bar{\varepsilon}\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\Phi+g_1\left(\frac{\partial}{\partial z}\right)T+h_1\left(\frac{\partial}{\partial z}\right)\mu=0, \qquad (13)$$

$$\left(\frac{\partial^2}{\partial^2 x}+\bar{\lambda}\frac{\partial^2}{\partial^2 z}\right)T=0, \qquad (14)$$

$$\left(\frac{\partial^2}{\partial^2 x}+\bar{D}\frac{\partial^2}{\partial^2 z}\right)\mu=0, \qquad (15)$$

where

$$\delta_1=\frac{c_{55}}{c_{11}},\quad \delta_2=\frac{c_{13}+c_{55}}{c_{11}},\quad \delta_3=\frac{c_{33}}{c_{11}},\quad e_1=\frac{e_{31}+e_{15}}{e_{33}},\quad e_2=\frac{e_{15}}{e_{33}},\quad \bar{\varepsilon}_p=\frac{e_{33}\Phi_0\omega_1^*}{c_{11}v_1},\quad \bar{\varepsilon}_q=\frac{\varepsilon_{33}\Phi_0\omega_1^*}{e_{33}v_1},$$

$$r_1=\frac{\beta_1 T_0}{c_{11}},\quad r_3=\frac{\beta_3 T_0}{c_{11}},\quad q_3=\frac{b_3 v_1^2}{c_{11}},\ q_1=\frac{b_1 v_1^2}{c_{11}},\quad \varepsilon=\frac{\varepsilon_{11}}{\varepsilon_{33}},\quad g_1=\frac{p_3 T_0}{e_{33}},\quad h_1=\frac{b_3 v_1^2}{e_{33}},\quad \bar{\lambda}=\frac{\lambda_3}{\lambda_1},$$

$$\bar{D}=\frac{D_3}{D_1},\quad \Phi_0=\frac{v_1\beta_1 T_0}{\omega^* e_{33}}.$$

The equations (11)-(15) can be written as

$$D\{u,w,\Phi,T,\mu\}^t=0, \qquad (16)$$

where D is the differential operator matrix given by

$$\begin{bmatrix} \frac{\partial^2}{\partial x^2}+\delta_1\frac{\partial^2}{\partial z^2} & \delta_2\frac{\partial^2}{\partial x\partial z} & e_1\bar{\varepsilon}_p\frac{\partial^2}{\partial x\partial z} & -r_1\frac{\partial}{\partial x} & -q_1\frac{\partial}{\partial x} \\ \delta_2\frac{\partial^2}{\partial x\partial z} & \delta_1\frac{\partial^2}{\partial x^2}+\delta_3\frac{\partial^2}{\partial z^2} & \bar{\varepsilon}_p\left(e_2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial z} & -r_3\frac{\partial}{\partial z} & -q_3\frac{\partial}{\partial z} \\ e_1\frac{\partial^2}{\partial x\partial z} & e_2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial x^2} & -\bar{\varepsilon}_q\left(\bar{\varepsilon}\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right) & g_1\frac{\partial}{\partial z} & h_1\frac{\partial}{\partial z} \\ 0 & 0 & 0 & \left(\frac{\partial^2}{\partial x^2}+\bar{a}\frac{\partial^2}{\partial x^2}\right) & 0 \\ 0 & 0 & 0 & 0 & \left(\frac{\partial^2}{\partial x^2}+\bar{D}\frac{\partial^2}{\partial x^2}\right) \end{bmatrix}. \qquad (17)$$

Equation (16) is a homogeneous set of differential equations in u, w, Φ, T, μ.The general solution by the operator theory as follows

$$u=A_{i1}F,\qquad w=A_{i2}F,\qquad \Phi=A_{i3}F,\qquad T=A_{i4}F\qquad \mu=A_{i5}F\quad (i=1,2,3,4,5). \qquad (18)$$

The determinant of the matrix D is given as

$$|D|=\left(a\frac{\partial^6}{\partial z^6}+b\frac{\partial^6}{\partial x^2\partial z^4}+c\frac{\partial^6}{\partial x^4\partial z^2}+d\frac{\partial^6}{\partial x^6}\right)\times\left(\frac{\partial^2}{\partial x^2}+\bar{\lambda}\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial x^2}+\bar{D}\frac{\partial^2}{\partial z^2}\right), \qquad (19)$$

where a, b, c, d are given in Appendix A. The function F in equation (18) satisfies the following homogeneous equation

$$\left|D\right| F = 0. \tag{20}$$

It can be seen that if i was set to 1 or 2 in equation (18), one can get two sets of general solution with $P = 0,\ \ T = 0$ and $\mu = 0$, which are actually to those for pure elasticity; $i = 1, 2,$ and 3 correspondence to the solution for piezoelectric discussed by Ding and Liang [31]; $i = 4$ correspondence to the general solution W_1 (say) with $\mu = 0$ which is identical to that for piezothermoelaticity. Taking $i = 5$ correspondence to the general solution W_2 (say) with $T = 0$.

Since the linear nature of the piezothermoelastic diffusion theory adopting in this paper, follows the same procedure as adopted by Li et al. [32] superposing W_1 and W_2 lead to

$$u = \left(a_1 \frac{\partial^6}{\partial x^6} + b_1 \frac{\partial^6}{\partial x^4 \partial z^2} + c_1 \frac{\partial^6}{\partial x^2 \partial z^4} + d_1 \frac{\partial^6}{\partial z^6} \right) \frac{\partial F}{\partial x}, \tag{21a}$$

$$w = \left(a_2 \frac{\partial^6}{\partial x^6} + b_2 \frac{\partial^6}{\partial x^4 \partial z^2} + c_2 \frac{\partial^6}{\partial x^2 \partial z^4} + d_2 \frac{\partial^6}{\partial z^6} \right) \frac{\partial F}{\partial z}, \tag{21b}$$

$$\Phi = \left(a_3 \frac{\partial^6}{\partial x^6} + b_3 \frac{\partial^6}{\partial x^4 \partial z^2} + c_3 \frac{\partial^6}{\partial x^2 \partial z^4} + d_3 \frac{\partial^6}{\partial z^6} \right) \frac{\partial F}{\partial z}, \tag{21c}$$

$$T = \left(a_4 \frac{\partial^8}{\partial x^8} + b_4 \frac{\partial^8}{\partial x^6 \partial z^2} + c_4 \frac{\partial^8}{\partial x^4 \partial z^4} + d_4 \frac{\partial^8}{\partial x^6 \partial z^2} + l_5 \frac{\partial^8}{\partial z^8} \right) F, \tag{21d}$$

$$\mu = \left(a_5 \frac{\partial^8}{\partial x^8} + b_5 \frac{\partial^8}{\partial x^6 \partial z^2} + c_5 \frac{\partial^8}{\partial x^4 \partial z^4} + d_5 \frac{\partial^8}{\partial x^6 \partial z^2} + l_6 \frac{\partial^8}{\partial z^8} \right) F, \tag{21e}$$

where the coefficients $a_k, b_k, c_k, d_k\ (k = 1, 2, 3, 4, 5)$ *and* l_5, l_6 are the expression given in Appendix B.
The general solutions of equations of (16) in terms of F can be rewritten as

$$\prod_{j=1}^{5} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z_j^2} \right) F = 0, \tag{22}$$

where $z_j = s_j z,\ \ s_4 = \sqrt{\lambda_1 / \lambda_3},\ \ s_5 = \sqrt{D_1 / D_3}$ and $s_j\ (j = 1, 2, 3)$ are three roots (with positive real part) of the following algebraic equation

$$as^6 - bs^4 + cs^2 - d = 0, \tag{23}$$

As known from the generalized Almansi (proved Ding et al. [30]) theorem, the function F can be expressed in terms of five harmonic functions

1. $F = F_1 + F_2 + F_3 + F_4 + F_5$ for distinct $s_j\ (j = 1, 2, 3, 4, 5)$,

2. $F = F_1 + F_2 + F_3 + F_4 + zF_5$ $\quad s_1 \neq s_2 \neq s_3 \neq s_4 = s_5$,

3. $F = F_1 + F_2 + F_3 + zF_4 + z^2 F_5$ $\quad$ for $s_1 \neq s_2 \neq s_3 = s_4 = s_5$,

4. $F = F_1 + F_2 + zF_3 + z^2 F_4 + z^3 F_4$ $\quad$ for $s_1 \neq s_2 = s_3 = s_4 = s_5$,

5. $F = F_1 + zF_2 + z^2 F_3 + z^3 F_4 + z^4 F_5$ $\quad$ for $s_1 = s_2 = s_3 = s_4 = s_5$, (24)

where F_j satisfies the following harmonic equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z_j^2}\right) F_j = 0 \qquad (j = 1, 2, 3, 4, 5). \tag{25}$$

The general solution for the case of distinct roots, can be derived as follows

$$u = \sum_{j=1}^{5} p_{1j} \frac{\partial^7 F_j}{\partial x \partial z_j^6},\ w = \sum_{j=1}^{5} s_j p_{2j} \frac{\partial^7 F_j}{\partial z_j^7},\ \ \Phi = \sum_{j=1}^{5} s_j p_{3j} \frac{\partial^7 F_j}{\partial z_j^7}, T = p_{44} \frac{\partial^8 F_4}{\partial z_4^8}, \mu = p_{55} \frac{\partial^8 F_5}{\partial z_5^8}. \tag{26}$$

Equation (26) can be further simplified by taking

$$p_{1j} \frac{\partial^6 F_j}{\partial z_j^6} = \psi_j. \tag{27}$$

Making use of (27) in equation (26) we yield

$$u = \sum_{j=1}^{5} \frac{\partial \psi_j}{\partial x}, \quad w = \sum_{j=1}^{5} s_j P_{1j} \frac{\partial \psi_j}{\partial z_j},\ \Phi = \sum_{j=1}^{5} s_j P_{1j} \frac{\partial \psi_j}{\partial z_j},\ T = P_{34} \frac{\partial^2 \psi_4}{\partial z_4^2}, C = P_{45} \frac{\partial^2 \psi_j}{\partial z_j^2}, \tag{28}$$

where $P_{1j} = p_{2j}/p_{1j}$, $\quad P_{2j} = p_{3j}/p_{1j}$, $\quad P_{34} = p_{44}/p_{14}$, $\quad P_{45} = p_{55}/p_{15}$.
The function ψ_j satisfies the harmonic equations

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z_j^2}\right) \Psi_j = 0. \quad (j = 1,2,3,4,5) \tag{29}$$

Applying the dimensionless quantities defined by (7) on equations (1)-(5), after suppressing the primes, with the aid of (28), we obtain

$$\sigma_{xx} = \sum_{j=1}^{5} \left(- f_1 + f_2 s_j^2 P_{1j} + f_3 s_j^2 P_{2j} - P_{3j} - f_4 P_{4j}\right) \frac{\partial^2 \psi_j}{\partial z_j^2}, \tag{30a}$$

$$\sigma_{zz} = \sum_{j=1}^{5} \left(- f_2 + f_5 s_j^2 P_{1j} + f_6 s_j^2 P_{2j} - f_7 P_{3j} - f_8 P_{4j}\right) \frac{\partial^2 \psi_j}{\partial z_j^2}, \tag{30b}$$

$$\sigma_{zx} = \sum_{j=1}^{5} \left[f_9 (1 + P_{1j}) + f_{10} P_{2j} \right] s_j \frac{\partial^2 \psi_j}{\partial x \partial z_j}, \tag{30c}$$

$$D_x = \sum_{j=1}^{5}\left[l_1(1+P_{1j}) - n_{10}P_{2j}\right]s_j \frac{\partial^2\psi_j}{\partial x\partial z_j}, \tag{30d}$$

$$D_z = \sum_{j=1}^{5}\left(-l_2 + l_3 s_j^2 P_{1j} - n_2 s_j^2 P_{2j} + n_3 P_{3j} - n_4 P_{4j}\right)\frac{\partial^2\psi_j}{\partial z_j^2}, \tag{30e}$$

where $P_{31} = P_{32} = P_{33} = P_{35} = 0$ and $P_{41} = P_{42} = P_{43} = P_{44} = 0,$ and

$f_1 = \dfrac{c_{11}}{\beta_1 T_0}$, $f_2 = \dfrac{c_{13}}{\beta_1 T_0}$, $f_3 = \dfrac{e_{31}\omega_1^*\Phi_0}{\beta_1 T_0 v_1}$, $f_4 = \dfrac{b_1\mu_0}{\beta_1 T_0}$ $f_5 = \dfrac{c_{33}}{b_1 T_0}$ $f_6 = \dfrac{e_{33}\omega_1^*\Phi_0}{\beta_1 T_0 v_1}$ $f_7 = \dfrac{\beta_3}{\beta_1}$, $f_8 = \dfrac{b_3\mu_0}{\beta_1 T_0}$,

$f_9 = \dfrac{c_{55}}{\beta_1 T_0}$, $f_{10} = \dfrac{e_{15}\omega_1^*\Phi_0}{\beta_1 T_0 v_1}$, $l_1 = \dfrac{e_{15}}{\sqrt{\beta_1 T_0}}$, $l_2 = \dfrac{e_{31}}{\sqrt{\beta_1 T_0}}$, $l_3 = \dfrac{e_{33}}{\sqrt{\beta_1 T_0}}$, $n_1 = \dfrac{\varepsilon_{11}\omega_1^*\Phi_0}{\sqrt{\beta_1 T_0}v_1}$, $n_2 = \dfrac{\varepsilon_{33}\omega_1^*\Phi_0}{\sqrt{\beta_1 T_0}v_1}$
,
$n_3 = \dfrac{P_3 T_0}{\sqrt{\beta_1 T_0}}$, $n_4 = \dfrac{b_3^*\mu_0}{\sqrt{\beta_1 T_0}}$.

Substituting equations (30) into equations (1)-(5), with the aid of (5)-(6) gives

$$f_1 - (f_2 P_{1j} + f_3 P_{2j})s_j^2 + P_{3j} + f_4 P_{4j} = \left[f_9(1+P_{1j}) + f_{10}P_{2j}\right]s_j^2,$$

$$-f_2 + (f_5 P_{1j} + f_6 P_{2j})s_j^2 - f_7 P_{3j} - f_8 P_{4j} = \left[f_9(1+P_{1j}) + f_{10}P_{2j}\right],$$

$$-l_2 + (l_3 P_{1j} - n_2 P_{2j})s_j^2 + n_3 P_{3j} + n_4 P_{4j} = l_1(1+P_{1j}),$$

$$\left(\lambda_1 - \lambda_3 s_j^2\right)P_{3j} = 0,$$

$$\left(D_1 - D_3 s_j^2\right)P_{4j} = 0. \qquad (j = 1,2,3,4,5) \tag{31}$$

By virtue of the above equations, the general solution (30) can be simplified as

$$\sigma_{xx} = -\sum_{j=1}^{5} s_j^2 w_{1j}\frac{\partial^2\psi_j}{\partial z_j^2}, \quad \sigma_{zz} = \sum_{j=1}^{5} w_{1j}\frac{\partial^2\psi_j}{\partial z_j^2}, \quad \sigma_{zx} = \sum_{j=1}^{5} s_j w_{1j}\frac{\partial^2\psi_j}{\partial x\partial z_j},$$

$$D_x = \sum_{j=1}^{5} s_j w_{2j}\frac{\partial^2\psi_j}{\partial x\partial z_j}, D_z = \sum_{j=1}^{5} w_{2j}\frac{\partial^2\psi_j}{\partial z_j^2}, \tag{32}$$

where

$$w_{1j} = \frac{f_1 - (f_2 P_{1j} + f_3 P_{2j})s_j^2 + P_{3j} + f_4 P_{4j}}{s_j^2} = f_9(1+P_{1j}) + f_{10}P_{2j} =$$

$$= -f_2 + (f_5 P_{1j} + f_6 P_{2j})s_j^2 - f_7 P_{3j} - f_8 P_{4j},$$

$$w_{2j} = -l_2 + (l_3 P_{1j} - n_2 P_{2j})s_j^2 + n_3 P_{3j} + n_4 P_{4j} = l_1(1+P_{1j}) - n_1 P_{2j}. \tag{33}$$

4. Fundamental solution for a point heat source and chemical potential source on the surface of a semi-infinite piezothermoelastic diffusion material

As shown in Fig. 1 we consider an orthotropic semi-infinite piezothermoelastic diffusion plane $z \geq 0$, whose isotropic plane is perpendicular to the z-axis. A point heat and chemical potential source H and P is applied at the origin of the two-dimensional Cartesian coordinate (x, z) and the surface z=0 is free, electro-thermally insulated and impermeable boundary. In Cartesian coordinate system, the general solution given by equations (28) and (32) in this semi-infinite plane is derived in this section.

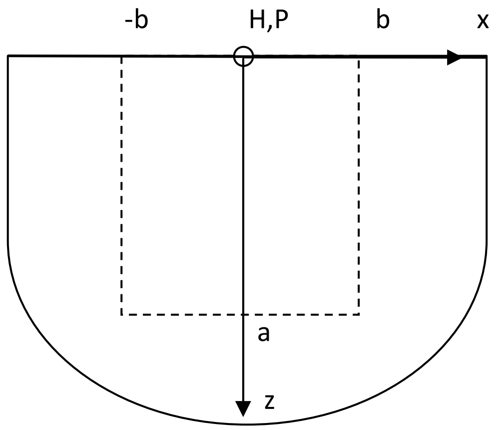

Fig. 1. A semi-infinite orthotropic piezothermoelastic diffusive plane with a point heat source H and chemical source P on the surface.

Introducing the harmonic functions

$$\Psi_j = A_j\left[\frac{1}{2}(z_j^2 - x^2)\left(\log r_j - \frac{3}{2}\right) - xz_j \tan^{-1}\frac{x}{z_j}\right], \qquad j = 1,2,3,4,5 \tag{34}$$

where $A_j\,(j = 1,2,3,4,5)$ are arbitrary constants and to be determined, and

$$r_j = \sqrt{x^2 + z_j^2}, \quad (j = 1, 2, 3, 4, 5) \tag{35}$$

The boundary conditions on the surface $z = 0$ are in the form of

$$\sigma_{zz} = \sigma_{zx} = 0, \qquad D_z = 0, \frac{\partial T}{\partial z} = 0, \qquad \frac{\partial \mu}{\partial z} = 0. \tag{36}$$

Considering the mechanical, electric, thermal equilibrium and chemical potential per unit mass for a rectangle of $0 \leq z \leq a$ and $-b \leq x \leq b\,(b > 0)$ (Fig. 1) four equations can be obtained

$$\int_{-b}^{b}\sigma_{zz}(x,a)dx + \int_0^a[\sigma_{zx}(b,z) - \sigma_{zx}(-b,z)]dz = 0, \tag{37a}$$

$$\int_{-b}^{b}D_z(x,a)dx + \int_0^a[D_x(b,z) - D_x(-b,z)]dz = 0, \tag{37b}$$

$$-\bar{\lambda}\int_{-b}^{b}\frac{\partial T}{\partial z}(x,a)dx-\int_{0}^{a}\left[\frac{\partial T}{\partial x}(b,z)-\frac{\partial T}{\partial x}(-b,z)\right]dz=H, \tag{37c}$$

$$-\bar{D}\int_{-b}^{b}\frac{\partial \mu}{\partial z}(x,a)dx-\int_{0}^{a}\left[\frac{\partial \mu}{\partial x}(b,z)-\frac{\partial \mu}{\partial x}(-b,z)\right]dz=P. \tag{37d}$$

Substituting equation (30) in equation (28) and (32), we obtain

$$u=-\sum_{j=1}^{5}A_j\left[x(\log r_j-1)+z_j\tan^{-1}\frac{x}{z_j}\right], \tag{38a}$$

$$w=\sum_{j=1}^{5}s_jP_{1j}A_j\left[z_j(\log r_j-1)-x\tan^{-1}\frac{x}{z_j}\right], \tag{38b}$$

$$\Phi=\sum_{j=1}^{5}s_jP_{2j}A_j\left[z_j(\log r_j-1)-x\tan^{-1}\frac{x}{z_j}\right], \tag{38c}$$

$$T=A_4P_{34}\log r_4, \tag{38d}$$

$$\mu=A_5P_{45}\log r_5, \tag{38e}$$

$$\sigma_{xx}=-\sum_{j=1}^{5}s_j^2w_{1j}A_j\log r_j, \tag{38f}$$

$$\sigma_{zz}=\sum_{j=1}^{5}w_{1j}A_j\log r_j, \tag{38g}$$

$$\sigma_{zx}=-\sum_{j=1}^{5}s_jw_{1j}A_j\tan^{-1}\frac{x}{z_j}, \tag{38h}$$

$$D_x=-\sum_{j=1}^{5}s_jw_{1j}A_j\tan^{-1}\frac{x}{z_j}, \tag{38i}$$

$$D_z=\sum_{j=1}^{5}w_{1j}A_j\log r_j. \tag{38j}$$

Substituting equation (38) in the given boundary conditions (36), we obtain

$$\sum_{j=1}^{5}w_{mj}A_j=0, \qquad (m=1,2) \tag{39}$$

$$\sum_{j=1}^{5}s_jw_{1j}A_j=0. \tag{40}$$

$\frac{\partial T}{\partial z}$ and $\frac{\partial \mu}{\partial z}$ are satisfied automatically.

Making use of (38f,g,h,i,j) into equation (37a,b) we yield

$$\sum_{j=1}^{4} w_{mj} A_j I_3 = 0. \quad (m = 1, 2), \tag{41}$$

where

$$I_3 = \left[x\left(\log\sqrt{x^2 + s_j^2 a^2} - 1\right) + s_j a \tan^{-1}\frac{x}{s_j a} \right]_{x=-b}^{x=b} - 2\left[z_j \tan^{-1}\frac{b}{z_j} + b\log\sqrt{b^2 + z_j^2} \right]_{z=0}^{z=a} = 2b(\log b - 1). \tag{42}$$

By virtue of the equation (42), equation (41) can be simplified to equation (39), i.e. equations (41) and (37 a, b) are satisfied automatically.

Some useful integrals are listed as follows:

$$\int \frac{\partial T}{\partial z} dx = s_4 P_{34} A_4 \int \frac{z_4}{r_4^2} dx = s_4 P_{34} A_4 \tan^{-1}\frac{x}{z_4}, \tag{43a}$$

$$\int \frac{\partial T}{\partial x} dz = P_{34} A_4 \int \frac{x}{r_4^2} dz = -\frac{P_{34}}{s_4} A_4 \tan^{-1}\frac{x}{z_4}, \tag{43b}$$

$$\int \frac{\partial \mu}{\partial z} dx = \sum_{j=1}^{4} A_j s_j^2 P_{2j} \int \frac{z}{x^2 + s_j^2 z^2} dx = \sum_{j=1}^{4} A_j s_j P_{2j} \tan^{-1}\frac{x}{s_j z}, \tag{43c}$$

$$\int \frac{\partial \mu}{\partial x} dz = \sum_{j=1}^{4} A_j P_{2j} \int \frac{x}{x^2 + s_j^2 z^2} dz = -\sum_{j=1}^{4} \frac{A_j}{s_j} P_{2j} \tan^{-1}\frac{x}{z_j}, \tag{43d}$$

Substituting equation (38d) into equation (37c) with using $s_4 = \sqrt{\lambda_1/\lambda_3}$ and the integrals (43a,b), we obtain

$$A_4 I_4 = \frac{H}{P_{34}\sqrt{\lambda_3/\lambda_1}}, \tag{44}$$

where

$$I_4 = -\left[\tan^{-1}\frac{x}{s_4 a} \right]_{x=-b}^{x=b} + \left[\tan^{-1}\frac{b}{z_4} \right]_{z=0}^{z=a} = -\pi. \tag{45}$$

A_4 can be determined by equations (44) and (45) as follows:

$$A_4 = -\frac{H}{\pi P_{34}\sqrt{\lambda_3/\lambda_1}}. \tag{46}$$

Substituting equation (38 b) into equation (37d) with using $s_5 = \sqrt{D_1/D_3}$ and the integrals (43c, d), we obtain

$$A_5 I_5 = \frac{P}{P_{45}\sqrt{D_3 / D_1}}, \tag{47}$$

where

$$I_5 = -\left[\tan^{-1}\frac{x}{s_5 a}\right]_{x=-b}^{x=b} + \left[\tan^{-1}\frac{b}{z_5}\right]_{z=0}^{z=a} = -\pi. \tag{48}$$

A_5 can be determined by equations (47) and (48), as follows:

$$A_5 = -\frac{P}{\pi P_{45}\sqrt{D_3 / D_1}}. \tag{49}$$

The five constants A_j $(j = 1, 2, 3, 4, 5)$ can be computed by five equations including equations (39), (40), (46), and (49).

Special case. In the absence of diffusion (38a)-(38j) and (39)-(40) reduce to

$$u = -\sum_{j=1}^{4} A_j \left[x(\log r_j - 1) + z_j \tan^{-1}\frac{x}{z_j} \right], \tag{50a}$$

$$w = \sum_{j=1}^{4} s_j P_{1j} A_j \left[z_j (\log r_j - 1) - x \tan^{-1}\frac{x}{z_j} \right], \tag{50b}$$

$$\Phi = \sum_{j=1}^{4} s_j P_{2j} A_j \left[z_j (\log r_j - 1) - x \tan^{-1}\frac{x}{z_j} \right], \tag{50c}$$

$$T = A_4 P_{34} \log r_4, \tag{50d}$$

$$\sigma_{xx} = -\sum_{j=1}^{4} s_j^2 w_{1j} A_j \log r_j, \tag{50e}$$

$$\sigma_{zz} = \sum_{j=1}^{4} w_{1j} A_j \log r_j, \tag{50f}$$

$$\sigma_{zx} = -\sum_{j=1}^{4} s_j w_{1j} A_j \tan^{-1}\frac{x}{z_j}, \tag{50g}$$

$$D_x = -\sum_{j=1}^{4} s_j w_{1j} A_j \tan^{-1}\frac{x}{z_j}, \tag{50h}$$

$$D_z = \sum_{j=1}^{4} w_{1j} A_j \log r_j, \tag{50i}$$

$$\sum_{j=1}^{4} w_{mj} A_j = 0, \qquad (m = 1, 2) \tag{51}$$

$$\sum_{j=1}^{4} s_j w_{1j} A_j = 0 . \tag{52}$$

$\frac{\partial T}{\partial z}$ is satisfied automatically. Also, from equation (46)

$$A_4 = -\frac{H}{\pi P_{34}\sqrt{\lambda_3/\lambda_1}} . \tag{53}$$

The above results are similar as obtained by Hou [20].
The four constants A_j $(j = 1, 2, 3, 4)$ can be determined by four equations including equations (51)-(53).

6. Numerical results and discussion

In order to determine the constants A_j $(j = 1, 2, 3, 4, 5)$, the method of Crammer's rule has been used to solve the system of non-homogeneous equations. We have used the MATLAB 7.04 software to computing the values of A_j $(j = 1, 2, 3, 4, 5)$ for computer programming.

The material chosen for numerical calculations is Cadmium Selenide (Cdse), which is orthotropic material. The physical data for piezothermoelastic as given by Sharma [33]:
$c_{11} = 74.1\times10^9 N\,m^{-2}$, $c_{12} = 45.2\times10^9 N\,m^{-2}$, $c_{13} = 39.3\times10^9 N\,m^{-2}$, $c_{33} = 83.6\times10^9 N\,m^{-2}$, $c_{55} = 13.2\times10^9 N\,m^{-2}$, $T_0 = 298\,K$, $\beta_1 = 6.21\times10^5\, C^2/Nm^2$, $\beta_3 = 5.51\times10^5\, C^2/Nm^2$, $\lambda_1 = 9W\,m^{-1}K^{-1}$, $\lambda_3 = 7W\,m^{-1}K^{-1}$, $e_{13} = -0.160\times10^{-3} cm^{-2}$, $e_{33} = 34\times10^{-3} cm^{-2}$, $e_{15} = -0.138\times10^{-3} cm^{-2}$, $\varepsilon_{11} = 8.26\times10^{-11}\, N\,m^{-2}/K$, $\varepsilon_{33} = 9.03\times10^{-11}\, N\,m^{-2}/K$, $p_3 = -2.9\times10^{-6}\, cm^{-2}/K$, $D_1 = 0.15C\,m^{-2}$, $D_3 = 0.25C\,m^{-2}$.

Behaviour of components of displacements, temperature change and chemical potential per unit mass. Figures (2)-(5) show the variations of components of displacement (u_1, u_3), electric potential, temperature change (T) and chemical potential per unit mass w.r.t. distance *x*.

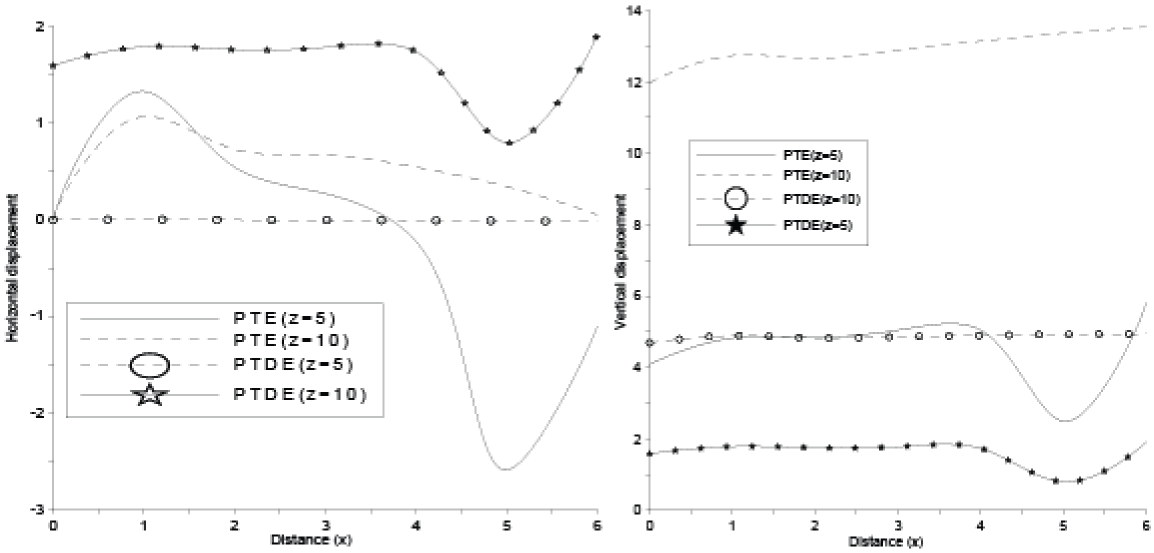

Fig. 2. Variation of (a) horizontal and (b) vertical displacements w.r.t. distance *x*.

The lines without center symbol correspond to piezothermelastic (PTE) and the centre symbol on these lines corresponds to piezothermoelastic diffusion (PTDE).

Figure 2a shows that the values of horizontal displacement u_1 for the cases of PTE (Z=5) and PTDE (Z=5) decrease with oscillation for smaller values of x and for higher values of *x*, it increase, but for PTDE (Z=10), the values of u_1 increase initially and decrease for higher values of x, moreover for the case of PTE (Z=10), the values of u_1 slightly increase. It is noticed that the values of u_1 in of PTE (Z=5) remain more (in comparison with PTE (z=10) and PTDE (z=5, z=10)).

Figure 2b shows that the values of vertical displacement u_3 for the cases of PTE (Z=10) and PTDE (Z=10) slightly increase for all values of x, but for PTE (Z=5) and PTDE (Z=5) the values of u_3 oscillates for smaller values of x and for higher values of x it increase. It is noticed that the values of u_3 in case of PTDE (Z=10) remain more (in comparison with PTE (z=5, z=10) and PTDE (z=5)).

Figure 3 shows that the values of electric displacement Φ decrease for the cases of PTE (Z=10) and PTDE (Z=10) for all values of x, whereas for the cases of PTE (Z=5) and PTDE (Z=5) the values of Φ decrease for smaller values of x, but for higher values of x, it slightly increases. It is observed that the values of Φ for the case of PTDE (z=10) remain more (in comparison with PTE (z=5, z=10) and PTDE (z=5)).

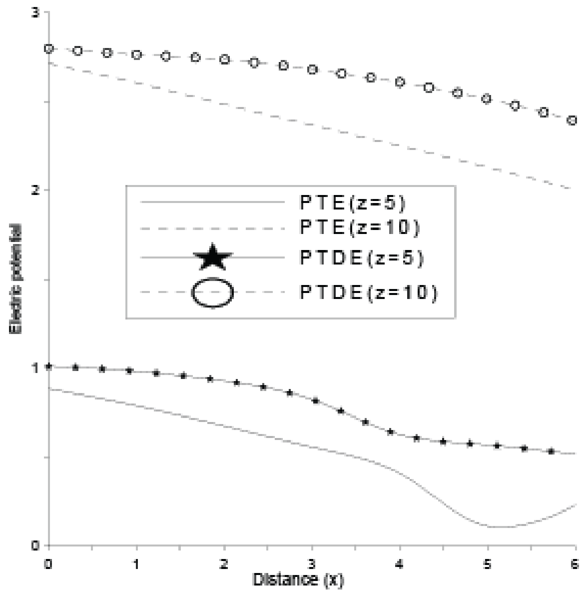

Fig. 3. Variation of electric potential w.r.t. distance *x*.

Figure 4 shows that the values of temperature change increase for both cases PTE (z=5, z=10) and for comparison it is noticed that the values of T remain more in case of PTE (z=10) (in comparison with PTE (z=5)) for all values of x.

Figure 5 shows that the values of chemical potential increase for both cases PTDE (z=5, z=10) and for comparison it is noticed that the values of chemical potential remain more in case of z=10 (in comparison with z=5).

7. Conclusions

The two-dimensional fundamental solution in orthotropic piezothermelastic diffusion media has been derived. Based on the obtained two-dimensional general solution in orthotropic piezothermoelastic diffusion media, the fundamental solution for a point heat source and a chemical potential source is obtained by five newly introduced harmonic functions ψ_j.

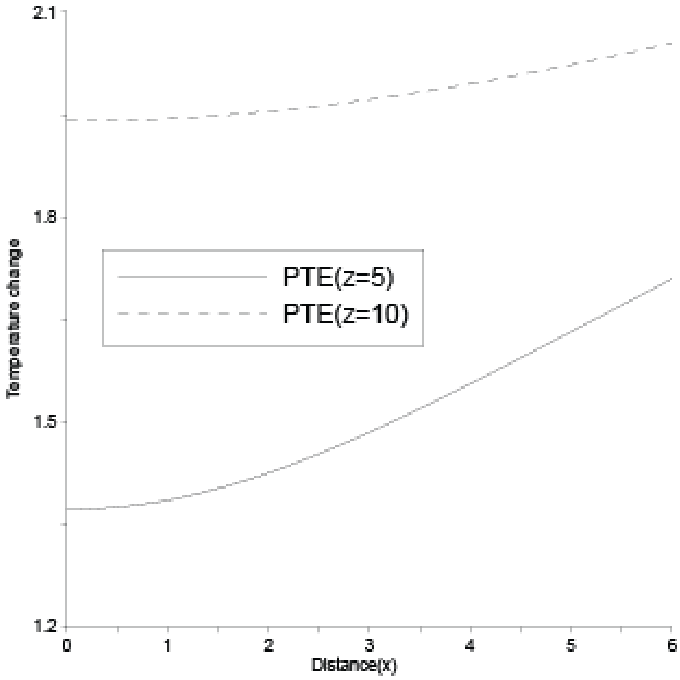

Fig. 4. Variation of temperature change (T) w.r.t. distance *x*.

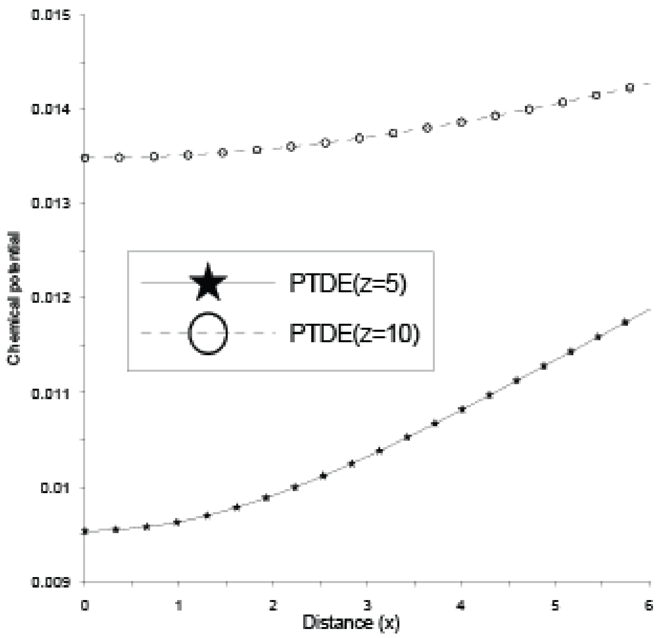

Fig. 5. Variation of chemical potential (P) w.r.t. distance *x*.

The components of displacement, electric potential, temperature change and chemical potential are computed numerically and presented graphically. From the present investigation, a special case of interest is also deduced to depict the effect of diffusion on components of displacement and electric potential. It is noticed that, the values of horizontal displacement (u_1) and electrical potential Φ in case of PTDE remain more (in comparison with PTE) but in case of vertical displacement (u_3) reverse behavior occurs. Significant diffusion effect is observed on components of displacement and temperature change.

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Appendix A

$$a = -\delta_1(\varepsilon_p + \delta_3),\;\; b = \varepsilon_q(\delta_2^2 - \delta_3) - \delta_1(\delta_1\varepsilon_q + \delta_3\bar{\varepsilon}) - 2\delta_1\varepsilon_p e_2 + 2\delta_2\varepsilon_p e_1 - \varepsilon_p(1 + e_1\delta_3\varepsilon_p)\,,$$

$$c = e_1\varepsilon_p(e_2\delta_2 - \delta_1 e_1) - \varepsilon_q(\delta_1 + \delta_3\bar{\varepsilon}) - 2\varepsilon_p e_2 - \delta_1\left(\delta_1\varepsilon_q\bar{\varepsilon} - \varepsilon_p e_2^2\right) + \delta_2(\delta_2\varepsilon_q\bar{\varepsilon} + e_1 e_2\varepsilon_p),$$

$$d = -(\varepsilon_p e_2^2 + \varepsilon_q\bar{\varepsilon}\delta_1)\,.$$

Appendix B

$$a_1 = -q_1(\varepsilon_q\delta_1\bar{\varepsilon} + \varepsilon_p e_2^2),$$

$$b_1 = \delta_2(\varepsilon_q q_3\bar{\varepsilon} - \varepsilon_p h_1 e_2) + \varepsilon_p e_1(\delta_1 h_1 + e_2 q_3) - \varepsilon_q q_1(\delta_1 + \delta_3\bar{\varepsilon}) - 2q_1 e_2\varepsilon_p - \bar{a}q_1(\varepsilon_q\delta_1\bar{\varepsilon} + \varepsilon_p e_2^2) +$$

$$c_1 = \delta_2(r_3\varepsilon_q - \varepsilon_p g_1) + e_1\varepsilon_p(g_1\delta_3 + r_3) - r_1(\varepsilon_q\delta_3 - \varepsilon_p) + \bar{D}[\delta_2\varepsilon_q(r_3\bar{\varepsilon} - g_1 e_2) + \varepsilon_p e_1(\delta_1 g_1 + e_2 r_3) +$$

$$+\delta_2(r_3\varepsilon_q\bar{\varepsilon} - \varepsilon_p g_1 e_2) + \varepsilon_p e_1(g_1\delta_1 + e_2 r_3) - \varepsilon_q r_1(\delta_1 + \delta_3\bar{\varepsilon}) - 2r_1 r_2\varepsilon_p - \bar{D}\varepsilon_p r_1(\delta_1\bar{\varepsilon} + e_2^2) -$$

$$-r_1(\varepsilon_q\delta_3 - \varepsilon_p) + \delta_2(q_3\varepsilon_q - h_1\varepsilon_p) + 2e_1\varepsilon_p h_1\delta_3 - q_1(\varepsilon_q\delta_3 + \varepsilon_p) + e_1 q_3\varepsilon_p] + a[\delta_2(q_3\varepsilon_p\bar{\varepsilon} - \varepsilon_p h_1 e_2) +$$

$$+\varepsilon_p e_1(\delta_1 h_1 + q_3 e_2) - 2q_1\varepsilon_p e_2 - \varepsilon_q(q_1\delta_1 + \delta_3 q\bar{\varepsilon})]\,,$$

$$d_1 = \bar{D}[\delta_2(\varepsilon_q r_3 - \varepsilon_p g_1) + \varepsilon_p e_1(g_1\delta_3 + r_3) - r_1(\varepsilon_q\delta_3 + \varepsilon_p)] + \bar{a}[\delta_2(\varepsilon_q q_3\bar{\varepsilon} - \varepsilon_p h_1 e_2) +$$

$$+\varepsilon_p e_1(\delta_1 h_1 + e_2 q_3) - \varepsilon_q q_1(\delta_1 + \delta_3\bar{\varepsilon}) - 2q_1\varepsilon_p e_2\,,$$

$$a_2 = \varepsilon_p e_2(g_1 + e_1 r_1) + \varepsilon_q\bar{\varepsilon}(r_1\delta_2 - r_3) + \varepsilon_p(h_1 e_2 - q_3\bar{\varepsilon}) + q_1(\varepsilon_q\delta_2\bar{\varepsilon} + e_1 e_2\varepsilon_p)\,,$$

$$b_2 = \bar{D}[\varepsilon_p e_2(g_1 + e_1 r_1) + \varepsilon_q\bar{\varepsilon}(r_1\delta_2 - r_3)] + \varepsilon_p g_1(1 + \delta_1 e_2) - \varepsilon_q r_3(1 + \delta_1\bar{\varepsilon}) - e_1(g_1\delta_2\varepsilon_p - e_1 r_3) +$$

$$+r_1(\varepsilon_q\delta_2 + e_1\varepsilon_p) + \bar{a}[\varepsilon_p(h_1 e_2 - q_3\bar{\varepsilon}) + q_1(\varepsilon_q\delta_2 + e_1 e_2\varepsilon_p)] + \varepsilon_p h_1(1 + \delta_1 e_2) - q_3(\varepsilon_p - \varepsilon_q\delta_1\bar{\varepsilon}) -$$

$$-e_1\varepsilon_p(h_1\delta_2 - e_1 q_3) + q_1(\varepsilon_q\delta_2 + e_1\varepsilon_p)\,,$$

$$c_2 = \bar{D}[\varepsilon_p g_1(1 + \delta_1 e_2) - \varepsilon_q r_3(1 + \delta_1\bar{\varepsilon})] + \delta_2(\varepsilon_q r_1 - g_1 e_1\varepsilon_p) + e_1(\varepsilon_p r_1 - e_1 r_3)] + \delta_1(\varepsilon_p g_1 - \varepsilon_q r_3) +$$

$$+\bar{a}[\varepsilon_p h_1(1 + \delta_1 e_2) - q_3(\varepsilon_p + \varepsilon_q\delta_1\bar{\varepsilon}) - e_1\varepsilon_p(\delta_2 h_1 + e_1 q_3) + q_1(\varepsilon_p e_1^2 + \varepsilon_q\delta_2)] + \varepsilon_p\delta_1(h_1 - q_3)\,,$$

$$d_2 = \bar{D}\,\delta_1(\varepsilon_p g_1 - \varepsilon_q r_3) + \bar{a}\,\delta_1(\varepsilon_p h_1 - \varepsilon_q q_3)\,,$$

$$a_3 = e_2(r_1\delta_2 - r_3) - \delta_1(g_1 + e_1 r_1) + e_2(q_1\delta_2 - q_3) - \delta_1(h_1 + q_1 e_1)\,,$$

$$b_3 = \bar{D}[e_2(r_1\delta_2 - r_3) - \delta_1(g_1 + e_1 r_1)]\delta_2(\delta_2 g_1 + r_3 e_1) + r_1(\delta_2 - \delta_3 e_1) + g_1(\delta_3 + \delta_1^2) + r_3(1 + \delta_1 e_2) +$$

$$+a[e_2(\delta_2 q_1 - q_3) - \delta_1(h_1 + q_1 e_1)] + h_1(\delta_2^2 - \delta_3) + q_3(\delta_2 e_1 - 1) - \delta_1(\delta_1 h_1 + q_3 e_2) + q_1(\delta_2 - \delta_3 e_1)\,,$$

$$c_3 = \bar{D}[\delta_2(\delta_2 g_1 + r_3 e_1) + r_1(\delta_2 - \delta_3 e_1) - g_1(\delta_3 + \delta_1^2) - r_3(1 + \delta_1 e_2)] - \delta_1(r_3 + \delta_3 g_1) +$$

$$+a[h_1(\delta_2^2 - \delta_3) - q_3(1 + \delta_1 e_2) - \delta_1^2 h_1 + q_3 e_1\delta_2 + q_1(\delta_2 - \delta_3 e_1)] - \delta_1(\delta_3 h_1 + q_3)\,,$$

$d_3 = -\bar{D}\delta_1(r_3 + \delta_3 g_1) - \bar{a}\delta_1(\delta_3 h_1 + q_3)$, $a_4 = d$, $b_4 = c + \bar{a}D$, $c_4 = b + c\bar{D}$, $d_4 = a + b\bar{D}$, $l_5 = a\bar{D}$,

$a_5 = d$, $b_5 = c + \bar{a}d$, $c_5 = b + \bar{a}c$, $d_5 = a + \bar{a}b$, $l_6 = a\bar{a}$.

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