

## FUNDAMENTAL SOLUTION FOR TWO-DIMENSIONAL PROBLEM IN ORTHOTROPIC PIEZOTHERMOELASTIC DIFFUSION MEDIA

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**Abstract.** The present investigation deals with the study of two-dimensional fundamental solution in orthotropic piezothermoelastic diffusion media. By virtue of the two-dimensional general solution of orthotropic piezothermodiffusion elastic media, the fundamental solution for a point heat source and chemical potential source on the surface of a semi-infinite orthotropic piezothermoelastic diffusion plane is constructed by five newly introduced harmonic functions. The components of displacement, stress, electric displacement, electric potential, temperature change and chemical potential are expressed in terms of elementary functions. The components of displacement, electric potential, temperature change and chemical potential are computed numerically and depicted graphically. From the present investigation, a special case of interest is also deduced to depict the effect of diffusion.

### 1. Introduction

Fundamental solutions or Green's functions play an important role in the solution of numerous problems in the mechanics and physics of solids. They are a basic building of the block of many further works. For example, fundamental solution can be used to construct many analytical solutions of practical problems, when boundary conditions are imposed. They are also very important in the boundary element method as well as the study of cracks, defects and inclusions.

Many studies have been directed towards obtaining full space Green's function for elastic solid in isotropic and anisotropic media due to Lord Kelvin [1], Freedholm [2], Synge [3], Pan and Chou [4]. Fourier transform method has been applied to investigate the Green's functions by Lifshitz and Rezentsveig [5] and Lejcek [6]. For anisotropic piezoelectric media, one can refer to important work of Deeg [7], Wang [8], Benveniste [9], and Chen and Lin [10]. Lee and Jiang [11] studied the boundary integral formulation and two-dimensional fundamental solution for piezoelectric media. Wang and Zheng [12] investigated the general solution for three-dimensional problem in piezoelectric media. Ding et al. [13] investigated the Fundamental solution for piezoelectric media. Ding et al. [14] studied the Fundamental solution for plane problem of piezoelectric materials.

The thermal effect is not considered in the above works. Rao and Sunar [15] pointed out the temperature variation in the piezoelectric media. Qin [16] derived the two-dimensional Green's functions of anisotropic pyroelectric media with holes of various shapes. Chen et al. [17] derived the general solution for transversely isotropic piezothermoelastic media. In this general solution all components of the coupled field are expressed by four harmonic functions. On the basis of this general solution Chen et al. [18] obtained Green's function of transversely isotropic pyroelectric media with a penny shaped. Hou et al. [19]



$$D_z = e_{31} \frac{\partial u}{\partial x} + e_{33} \frac{\partial w}{\partial z} - \varepsilon_{33} \frac{\partial \Phi}{\partial z} + p_3 T, \quad (5)$$

where  $u$  and  $w$  are components of the mechanical displacement in  $x$  and  $z$  directions, respectively;  $\sigma_{ij}$  and  $D_i$  are the components of stress and electric displacement, respectively;  $\beta_i$  and  $b_i$  are material constants.  $\Phi$  and  $T$  are electric potential and temperature increment, respectively;  $c_{ij}$ ,  $e_{ij}$ ,  $\varepsilon_{ij}$ , and  $p_3$  are elastic piezoelectric, dielectric, thermal modules, diffusion modules and pyroelectric constants, respectively.

The mechanical, electric, heat equilibrium and mass diffusions equations for static problem, in the absence of body forces, free charges, heat sources and mass diffusive sources are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial z} = 0, \quad \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = 0, \quad (6)$$

$$\frac{\partial D_x}{\partial x} + \frac{\partial \sigma_z}{\partial z} = 0, \quad (7)$$

$$\left( \lambda_1 \frac{\partial^2}{\partial^2 x} + \lambda_3 \frac{\partial^2}{\partial^2 z} \right) T = 0, \quad (8)$$

$$\left( D_1 \frac{\partial^2}{\partial^2 x} + D_3 \frac{\partial^2}{\partial^2 z} \right) \mu = 0. \quad (9)$$

We define the dimensionless quantities:

$$x' = \frac{\omega_1^* x_i}{v_1}, \quad z' = \frac{\omega_1^* z}{v_1}, \quad u' = \frac{\omega_1^* u}{v_1}, \quad w' = \frac{\omega_1^* w}{v_1}, \quad T' = \frac{T}{T_0}, \quad \Phi' = \frac{\omega^* e_{33} \Phi}{v_1 \beta_1 T_0}, \quad \mu' = \frac{\mu}{v_1^2},$$

$$\sigma'_{ij} = \frac{\sigma_{ij}}{\beta_1 T_0}, \quad D'_i = \frac{D_i}{\sqrt{\beta_1 T_0}}, \quad H' = \frac{v_1}{T_0 \lambda_1 \omega_1^*} H, \quad P' = \frac{P}{D_1 v_1 \omega_1^*} H,$$

where

$$v_1^2 = \frac{\beta_1 T_0}{b_1}, \quad \omega_1^* = \frac{\beta_1 c_{11}}{b_1 \lambda_1}. \quad (10)$$

Substituting equations (1)-(5) into equations (6)-(7) and applying the dimensionless quantities defined by Eq. (10) on resulting equations, after suppressing the primes, we obtain

$$\left( \frac{\partial^2}{\partial x^2} + \delta_1 \frac{\partial^2}{\partial z^2} \right) u + \left( \delta_2 \frac{\partial^2}{\partial x \partial z} \right) w - e_1 \bar{\varepsilon}_p \frac{\partial^2 \Phi}{\partial x \partial z} - r_1 \left( \frac{\partial}{\partial x} \right) T - q_1 \left( \frac{\partial}{\partial x} \right) \mu = 0, \quad (11)$$

$$\left( \delta_2 \frac{\partial^2}{\partial x \partial z} \right) u + \left( \delta_1 \frac{\partial^2}{\partial x^2} + \delta_3 \frac{\partial^2}{\partial z^2} \right) w - \bar{\varepsilon}_p \left( e_2 \frac{\partial^2}{\partial x^2} + \delta_3 \frac{\partial^2}{\partial z^2} \right) \Phi - r_3 \left( \frac{\partial}{\partial z} \right) T - q_3 \left( \frac{\partial}{\partial z} \right) \mu = 0, \quad (12)$$



where  $a, b, c, d$  are given in Appendix A. The function  $F$  in equation (18) satisfies the following homogeneous equation

$$|D|F = 0. \quad (20)$$

It can be seen that if  $i$  was set to 1 or 2 in equation (18), one can get two sets of general solution with  $P=0$ ,  $T=0$  and  $\mu=0$ , which are actually to those for pure elasticity;  $i=1, 2$ , and 3 correspondence to the solution for piezoelectric discussed by Ding and Liang [31];  $i=4$  correspondence to the general solution  $W_1$  (say) with  $\mu=0$  which is identical to that for piezothermoelasticity. Taking  $i=5$  correspondence to the general solution  $W_2$  (say) with  $T=0$ .

Since the linear nature of the piezothermoelastic diffusion theory adopting in this paper, follows the same procedure as adopted by Li et al. [32] superposing  $W_1$  and  $W_2$  lead to

$$u = \left( a_1 \frac{\partial^6}{\partial x^6} + b_1 \frac{\partial^6}{\partial x^4 \partial z^2} + c_1 \frac{\partial^6}{\partial x^2 \partial z^4} + d_1 \frac{\partial^6}{\partial z^6} \right) \frac{\partial F}{\partial x}, \quad (21a)$$

$$w = \left( a_2 \frac{\partial^6}{\partial x^6} + b_2 \frac{\partial^6}{\partial x^4 \partial z^2} + c_2 \frac{\partial^6}{\partial x^2 \partial z^4} + d_2 \frac{\partial^6}{\partial z^6} \right) \frac{\partial F}{\partial z}, \quad (21b)$$

$$\Phi = \left( a_3 \frac{\partial^6}{\partial x^6} + b_3 \frac{\partial^6}{\partial x^4 \partial z^2} + c_3 \frac{\partial^6}{\partial x^2 \partial z^4} + d_3 \frac{\partial^6}{\partial z^6} \right) \frac{\partial F}{\partial z}, \quad (21c)$$

$$T = \left( a_4 \frac{\partial^8}{\partial x^8} + b_4 \frac{\partial^8}{\partial x^6 \partial z^2} + c_4 \frac{\partial^8}{\partial x^4 \partial z^4} + d_4 \frac{\partial^8}{\partial x^2 \partial z^6} + l_5 \frac{\partial^8}{\partial z^8} \right) F, \quad (21d)$$

$$\mu = \left( a_5 \frac{\partial^8}{\partial x^8} + b_5 \frac{\partial^8}{\partial x^6 \partial z^2} + c_5 \frac{\partial^8}{\partial x^4 \partial z^4} + d_5 \frac{\partial^8}{\partial x^2 \partial z^6} + l_6 \frac{\partial^8}{\partial z^8} \right) F, \quad (21e)$$

where the coefficients  $a_k, b_k, c_k, d_k$  ( $k=1, 2, 3, 4, 5$ ) and  $l_5, l_6$  are the expression given in Appendix B.

The general solutions of equations of (16) in terms of  $F$  can be rewritten as

$$\prod_{j=1}^5 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z_j^2} \right) F = 0, \quad (22)$$

where  $z_j = s_j z$ ,  $s_4 = \sqrt{\lambda_1 / \lambda_3}$ ,  $s_5 = \sqrt{D_1 / D_3}$  and  $s_j$  ( $j=1, 2, 3$ ) are three roots (with positive real part) of the following algebraic equation

$$as^6 - bs^4 + cs^2 - d = 0, \quad (23)$$

As known from the generalized Almansi (proved Ding et al. [30]) theorem, the function  $F$  can be expressed in terms of five harmonic functions

$$1. F = F_1 + F_2 + F_3 + F_4 + F_5 \quad \text{for distinct } s_j \ (j=1, 2, 3, 4, 5),$$



$$D_x = \sum_{j=1}^5 \left[ l_1(1+P_{1j}) - n_{10}P_{2j} \right] s_j \frac{\partial^2 \psi_j}{\partial x \partial z_j}, \quad (30d)$$

$$D_z = \sum_{j=1}^5 \left( -l_2 + l_3 s_j^2 P_{1j} - n_2 s_j^2 P_{2j} + n_3 P_{3j} - n_4 P_{4j} \right) \frac{\partial^2 \psi_j}{\partial z_j^2}, \quad (30e)$$

where  $P_{31} = P_{32} = P_{33} = P_{35} = 0$  and  $P_{41} = P_{42} = P_{43} = P_{44} = 0$ , and

$$f_1 = \frac{c_{11}}{\beta_1 T_0}, \quad f_2 = \frac{c_{13}}{\beta_1 T_0}, \quad f_3 = \frac{e_{31} \omega_1^* \Phi_0}{\beta_1 T_0 v_1}, \quad f_4 = \frac{b_1 \mu_0}{\beta_1 T_0}, \quad f_5 = \frac{c_{33}}{b_1 T_0}, \quad f_6 = \frac{e_{33} \omega_1^* \Phi_0}{\beta_1 T_0 v_1}, \quad f_7 = \frac{\beta_3}{\beta_1}, \quad f_8 = \frac{b_3 \mu_0}{\beta_1 T_0},$$

$$f_9 = \frac{c_{55}}{\beta_1 T_0}, \quad f_{10} = \frac{e_{15} \omega_1^* \Phi_0}{\beta_1 T_0 v_1}, \quad l_1 = \frac{e_{15}}{\sqrt{\beta_1 T_0}}, \quad l_2 = \frac{e_{31}}{\sqrt{\beta_1 T_0}}, \quad l_3 = \frac{e_{33}}{\sqrt{\beta_1 T_0}}, \quad n_1 = \frac{\varepsilon_{11} \omega_1^* \Phi_0}{\sqrt{\beta_1 T_0} v_1}, \quad n_2 = \frac{\varepsilon_{33} \omega_1^* \Phi_0}{\sqrt{\beta_1 T_0} v_1},$$

$$n_3 = \frac{P_3 T_0}{\sqrt{\beta_1 T_0}}, \quad n_4 = \frac{b_3^* \mu_0}{\sqrt{\beta_1 T_0}}.$$

Substituting equations (30) into equations (1)-(5), with the aid of (5)-(6) gives

$$f_1 - (f_2 P_{1j} + f_3 P_{2j}) s_j^2 + P_{3j} + f_4 P_{4j} = [f_9(1+P_{1j}) + f_{10} P_{2j}] s_j^2,$$

$$-f_2 + (f_5 P_{1j} + f_6 P_{2j}) s_j^2 - f_7 P_{3j} - f_8 P_{4j} = [f_9(1+P_{1j}) + f_{10} P_{2j}],$$

$$-l_2 + (l_3 P_{1j} - n_2 P_{2j}) s_j^2 + n_3 P_{3j} + n_4 P_{4j} = l_1(1+P_{1j}),$$

$$(\lambda_1 - \lambda_3 s_j^2) P_{3j} = 0,$$

$$(D_1 - D_3 s_j^2) P_{4j} = 0. \quad (j=1, 2, 3, 4, 5) \quad (31)$$

By virtue of the above equations, the general solution (30) can be simplified as

$$\sigma_{xx} = -\sum_{j=1}^5 s_j^2 w_{1j} \frac{\partial^2 \psi_j}{\partial z_j^2}, \quad \sigma_{zz} = \sum_{j=1}^5 w_{1j} \frac{\partial^2 \psi_j}{\partial z_j^2}, \quad \sigma_{xz} = \sum_{j=1}^5 s_j w_{1j} \frac{\partial^2 \psi_j}{\partial x \partial z_j},$$

$$D_x = \sum_{j=1}^5 s_j w_{2j} \frac{\partial^2 \psi_j}{\partial x \partial z_j}, \quad D_z = \sum_{j=1}^5 w_{2j} \frac{\partial^2 \psi_j}{\partial z_j^2}, \quad (32)$$

where

$$w_{1j} = \frac{f_1 - (f_2 P_{1j} + f_3 P_{2j}) s_j^2 + P_{3j} + f_4 P_{4j}}{s_j^2} = f_9(1+P_{1j}) + f_{10} P_{2j} =$$

$$= -f_2 + (f_5 P_{1j} + f_6 P_{2j}) s_j^2 - f_7 P_{3j} - f_8 P_{4j},$$

$$w_{2j} = -l_2 + (l_3 P_{1j} - n_2 P_{2j}) s_j^2 + n_3 P_{3j} + n_4 P_{4j} = l_1(1+P_{1j}) - n_1 P_{2j}. \quad (33)$$





$$-\bar{\lambda} \int_{-b}^b \frac{\partial T}{\partial z}(x, a) dx - \int_0^a \left[ \frac{\partial T}{\partial x}(b, z) - \frac{\partial T}{\partial x}(-b, z) \right] dz = H, \quad (37c)$$

$$-\bar{D} \int_{-b}^b \frac{\partial \mu}{\partial z}(x, a) dx - \int_0^a \left[ \frac{\partial \mu}{\partial x}(b, z) - \frac{\partial \mu}{\partial x}(-b, z) \right] dz = P. \quad (37d)$$

Substituting equation (30) in equation (28) and (32), we obtain

$$u = -\sum_{j=1}^5 A_j \left[ x(\log r_j - 1) + z_j \tan^{-1} \frac{x}{z_j} \right], \quad (38a)$$

$$w = \sum_{j=1}^5 s_j P_{1j} A_j \left[ z_j (\log r_j - 1) - x \tan^{-1} \frac{x}{z_j} \right], \quad (38b)$$

$$\Phi = \sum_{j=1}^5 s_j P_{2j} A_j \left[ z_j (\log r_j - 1) - x \tan^{-1} \frac{x}{z_j} \right], \quad (38c)$$

$$T = A_4 P_{34} \log r_4, \quad (38d)$$

$$\mu = A_5 P_{45} \log r_5, \quad (38e)$$

$$\sigma_{xx} = -\sum_{j=1}^5 s_j^2 w_{1j} A_j \log r_j, \quad (38f)$$

$$\sigma_{zz} = \sum_{j=1}^5 w_{1j} A_j \log r_j, \quad (38g)$$

$$\sigma_{zx} = -\sum_{j=1}^5 s_j w_{1j} A_j \tan^{-1} \frac{x}{z_j}, \quad (38h)$$

$$D_x = -\sum_{j=1}^5 s_j w_{1j} A_j \tan^{-1} \frac{x}{z_j}, \quad (38i)$$

$$D_z = \sum_{j=1}^5 w_{1j} A_j \log r_j. \quad (38j)$$

Substituting equation (38) in the given boundary conditions (36), we obtain

$$\sum_{j=1}^5 w_{mj} A_j = 0, \quad (m = 1, 2) \quad (39)$$

$$\sum_{j=1}^5 s_j w_{1j} A_j = 0. \quad (40)$$



$$A_5 I_5 = \frac{P}{P_{45} \sqrt{D_3 / D_1}}, \quad (47)$$

where

$$I_5 = - \left[ \tan^{-1} \frac{x}{s_5 a} \right]_{x=-b}^{x=b} + \left[ \tan^{-1} \frac{b}{z_5} \right]_{z=0}^{z=a} = -\pi. \quad (48)$$

$A_5$  can be determined by equations (47) and (48), as follows:

$$A_5 = - \frac{P}{\pi P_{45} \sqrt{D_3 / D_1}}. \quad (49)$$

The five constants  $A_j$  ( $j=1, 2, 3, 4, 5$ ) can be computed by five equations including equations (39), (40), (46), and (49).

**Special case.** In the absence of diffusion (38a)-(38j) and (39)-(40) reduce to

$$u = - \sum_{j=1}^4 A_j \left[ x(\log r_j - 1) + z_j \tan^{-1} \frac{x}{z_j} \right], \quad (50a)$$

$$w = \sum_{j=1}^4 s_j P_{1j} A_j \left[ z_j(\log r_j - 1) - x \tan^{-1} \frac{x}{z_j} \right], \quad (50b)$$

$$\Phi = \sum_{j=1}^4 s_j P_{2j} A_j \left[ z_j(\log r_j - 1) - x \tan^{-1} \frac{x}{z_j} \right], \quad (50c)$$

$$T = A_4 P_{34} \log r_4, \quad (50d)$$

$$\sigma_{xx} = - \sum_{j=1}^4 s_j^2 w_{1j} A_j \log r_j, \quad (50e)$$

$$\sigma_{zz} = \sum_{j=1}^4 w_{1j} A_j \log r_j, \quad (50f)$$

$$\sigma_{zx} = - \sum_{j=1}^4 s_j w_{1j} A_j \tan^{-1} \frac{x}{z_j}, \quad (50g)$$

$$D_x = - \sum_{j=1}^4 s_j w_{1j} A_j \tan^{-1} \frac{x}{z_j}, \quad (50h)$$

$$D_z = \sum_{j=1}^4 w_{1j} A_j \log r_j, \quad (50i)$$

$$\sum_{j=1}^4 w_{mj} A_j = 0, \quad (m=1, 2) \quad (51)$$

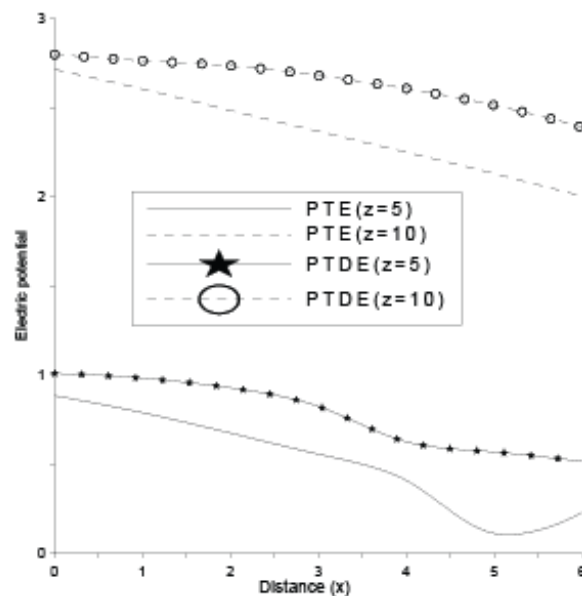


The lines without center symbol correspond to piezothermelastic (PTE) and the centre symbol on these lines corresponds to piezothermoelastic diffusion (PTDE).

Figure 2a shows that the values of horizontal displacement  $u_1$  for the cases of PTE ( $Z=5$ ) and PTDE ( $Z=5$ ) decrease with oscillation for smaller values of  $x$  and for higher values of  $x$ , it increase, but for PTDE ( $Z=10$ ), the values of  $u_1$  increase initially and decrease for higher values of  $x$ , moreover for the case of PTE ( $Z=10$ ), the values of  $u_1$  slightly increase. It is noticed that the values of  $u_1$  in of PTE ( $Z=5$ ) remain more (in comparison with PTE ( $z=10$ ) and PTDE ( $z=5, z=10$ )).

Figure 2b shows that the values of vertical displacement  $u_3$  for the cases of PTE ( $Z=10$ ) and PTDE ( $Z=10$ ) slightly increase for all values of  $x$ , but for PTE ( $Z=5$ ) and PTDE ( $Z=5$ ) the values of  $u_3$  oscillates for smaller values of  $x$  and for higher values of  $x$  it increase. It is noticed that the values of  $u_3$  in case of PTDE ( $Z=10$ ) remain more (in comparison with PTE ( $z=5, z=10$ ) and PTDE ( $z=5$ )).

Figure 3 shows that the values of electric displacement  $\Phi$  decrease for the cases of PTE ( $Z=10$ ) and PTDE ( $Z=10$ ) for all values of  $x$ , whereas for the cases of PTE ( $Z=5$ ) and PTDE ( $Z=5$ ) the values of  $\Phi$  decrease for smaller values of  $x$ , but for higher values of  $x$ , it slightly increases. It is observed that the values of  $\Phi$  for the case of PTDE ( $z=10$ ) remain more (in comparison with PTE ( $z=5, z=10$ ) and PTDE ( $z=5$ )).



**Fig. 3.** Variation of electric potential w.r.t. distance  $x$ .

Figure 4 shows that the values of temperature change increase for both cases PTE ( $z=5, z=10$ ) and for comparison it is noticed that the values of  $T$  remain more in case of PTE ( $z=10$ ) (in comparison with PTE ( $z=5$ )) for all values of  $x$ .

Figure 5 shows that the values of chemical potential increase for both cases PTDE ( $z=5, z=10$ ) and for comparison it is noticed that the values of chemical potential remain more in case of  $z=10$  (in comparison with  $z=5$ ).

## 7. Conclusions

The two-dimensional fundamental solution in orthotropic piezothermelastic diffusion media has been derived. Based on the obtained two-dimensional general solution in orthotropic piezothermoelastic diffusion media, the fundamental solution for a point heat source and a chemical potential source is obtained by five newly introduced harmonic functions  $\psi_j$ .



## Appendix A

$$\begin{aligned}
 a &= -\delta_1(\varepsilon_p + \delta_3), \quad b = \varepsilon_q(\delta_2^2 - \delta_3) - \delta_1(\delta_1\varepsilon_q + \delta_3\bar{\varepsilon}) - 2\delta_1\varepsilon_p e_2 + 2\delta_2\varepsilon_p e_1 - \varepsilon_p(1 + e_1\delta_3\varepsilon_p), \\
 c &= e_1\varepsilon_p(e_2\delta_2 - \delta_1e_1) - \varepsilon_q(\delta_1 + \delta_3\bar{\varepsilon}) - 2\varepsilon_p e_2 - \delta_1(\delta_1\varepsilon_q\bar{\varepsilon} - \varepsilon_p e_2^2) + \delta_2(\delta_2\varepsilon_q\bar{\varepsilon} + e_1e_2\varepsilon_p), \\
 d &= -(\varepsilon_p e_2^2 + \varepsilon_q\bar{\varepsilon}\delta_1).
 \end{aligned}$$

## Appendix B

$$\begin{aligned}
 a_1 &= -q_1(\varepsilon_q\delta_1\bar{\varepsilon} + \varepsilon_p e_2^2), \\
 b_1 &= \delta_2(\varepsilon_q q_3\bar{\varepsilon} - \varepsilon_p h_1 e_2) + \varepsilon_p e_1(\delta_1 h_1 + e_2 q_3) - \varepsilon_q q_1(\delta_1 + \delta_3\bar{\varepsilon}) - 2q_1 e_2 \varepsilon_p - \bar{a}q_1(\varepsilon_q\delta_1\bar{\varepsilon} + \varepsilon_p e_2^2) + \\
 c_1 &= \delta_2(r_3\varepsilon_q - \varepsilon_p g_1) + e_1\varepsilon_p(g_1\delta_3 + r_3) - r_1(\varepsilon_q\delta_3 - \varepsilon_p) + \bar{D}[\delta_2\varepsilon_q(r_3\bar{\varepsilon} - g_1 e_2) + \varepsilon_p e_1(\delta_1 g_1 + e_2 r_3) + \\
 &\quad + \delta_2(r_3\varepsilon_q\bar{\varepsilon} - \varepsilon_p g_1 e_2) + \varepsilon_p e_1(g_1\delta_1 + e_2 r_3) - \varepsilon_q r_1(\delta_1 + \delta_3\bar{\varepsilon}) - 2r_1 r_2 \varepsilon_p - \bar{D}\varepsilon_p r_1(\delta_1\bar{\varepsilon} + e_2^2) - \\
 &\quad - r_1(\varepsilon_q\delta_3 - \varepsilon_p) + \delta_2(q_3\varepsilon_q - h_1\varepsilon_p) + 2e_1\varepsilon_p h_1\delta_3 - q_1(\varepsilon_q\delta_3 + \varepsilon_p) + e_1 q_3 \varepsilon_p] + a[\delta_2(q_3\varepsilon_p\bar{\varepsilon} - \varepsilon_p h_1 e_2) + \\
 &\quad + \varepsilon_p e_1(\delta_1 h_1 + q_3 e_2) - 2q_1\varepsilon_p e_2 - \varepsilon_q(q_1\delta_1 + \delta_3 q\bar{\varepsilon})], \\
 d_1 &= \bar{D}[\delta_2(\varepsilon_q r_3 - \varepsilon_p g_1) + \varepsilon_p e_1(g_1\delta_3 + r_3) - r_1(\varepsilon_q\delta_3 + \varepsilon_p)] + \bar{a}[\delta_2(\varepsilon_q q_3\bar{\varepsilon} - \varepsilon_p h_1 e_2) + \\
 &\quad + \varepsilon_p e_1(\delta_1 h_1 + e_2 q_3) - \varepsilon_q q_1(\delta_1 + \delta_3\bar{\varepsilon}) - 2q_1\varepsilon_p e_2], \\
 a_2 &= \varepsilon_p e_2(g_1 + e_1 r_1) + \varepsilon_q\bar{\varepsilon}(r_1\delta_2 - r_3) + \varepsilon_p(h_1 e_2 - q_3\bar{\varepsilon}) + q_1(\varepsilon_q\delta_2\bar{\varepsilon} + e_1 e_2 \varepsilon_p), \\
 b_2 &= \bar{D}[\varepsilon_p e_2(g_1 + e_1 r_1) + \varepsilon_q\bar{\varepsilon}(r_1\delta_2 - r_3)] + \varepsilon_p g_1(1 + \delta_1 e_2) - \varepsilon_q r_3(1 + \delta_1\bar{\varepsilon}) - e_1(g_1\delta_2\varepsilon_p - e_1 r_3) + \\
 &\quad + r_1(\varepsilon_q\delta_2 + e_1\varepsilon_p) + \bar{a}[\varepsilon_p(h_1 e_2 - q_3\bar{\varepsilon}) + q_1(\varepsilon_q\delta_2 + e_1 e_2 \varepsilon_p)] + \varepsilon_p h_1(1 + \delta_1 e_2) - q_3(\varepsilon_p - \varepsilon_q\delta_1\bar{\varepsilon}) - \\
 &\quad - e_1\varepsilon_p(h_1\delta_2 - e_1 q_3) + q_1(\varepsilon_q\delta_2 + e_1\varepsilon_p), \\
 c_2 &= \bar{D}[\varepsilon_p g_1(1 + \delta_1 e_2) - \varepsilon_q r_3(1 + \delta_1\bar{\varepsilon})] + \delta_2(\varepsilon_q r_1 - g_1 e_1 \varepsilon_p) + e_1(\varepsilon_p r_1 - e_1 r_3) + \delta_1(\varepsilon_p g_1 - \varepsilon_q r_3) + \\
 &\quad + \bar{a}[\varepsilon_p h_1(1 + \delta_1 e_2) - q_3(\varepsilon_p + \varepsilon_q\delta_1\bar{\varepsilon}) - e_1\varepsilon_p(\delta_2 h_1 + e_1 q_3) + q_1(\varepsilon_p e_1^2 + \varepsilon_q\delta_2)] + \varepsilon_p\delta_1(h_1 - q_3), \\
 d_2 &= \bar{D}\delta_1(\varepsilon_p g_1 - \varepsilon_q r_3) + \bar{a}\delta_1(\varepsilon_p h_1 - \varepsilon_q q_3), \\
 a_3 &= e_2(r_1\delta_2 - r_3) - \delta_1(g_1 + e_1 r_1) + e_2(q_1\delta_2 - q_3) - \delta_1(h_1 + q_1 e_1), \\
 b_3 &= \bar{D}[e_2(r_1\delta_2 - r_3) - \delta_1(g_1 + e_1 r_1)]\delta_2(\delta_2 g_1 + r_3 e_1) + r_1(\delta_2 - \delta_3 e_1) + g_1(\delta_3 + \delta_1^2) + r_3(1 + \delta_1 e_2) + \\
 &\quad + a[e_2(\delta_2 q_1 - q_3) - \delta_1(h_1 + q_1 e_1)] + h_1(\delta_2^2 - \delta_3) + q_3(\delta_2 e_1 - 1) - \delta_1(\delta_1 h_1 + q_3 e_2) + q_1(\delta_2 - \delta_3 e_1), \\
 c_3 &= \bar{D}[\delta_2(\delta_2 g_1 + r_3 e_1) + r_1(\delta_2 - \delta_3 e_1) - g_1(\delta_3 + \delta_1^2) - r_3(1 + \delta_1 e_2)] - \delta_1(r_3 + \delta_3 g_1) + \\
 &\quad + a[h_1(\delta_2^2 - \delta_3) - q_3(1 + \delta_1 e_2) - \delta_1^2 h_1 + q_3 e_1 \delta_2 + q_1(\delta_2 - \delta_3 e_1)] - \delta_1(\delta_3 h_1 + q_3),
 \end{aligned}$$

$$d_3 = -\bar{D}\delta_1(r_3 + \delta_3 g_1) - \bar{a}\delta_1(\delta_3 h_1 + q_3), \quad a_4 = d, \quad b_4 = c + \bar{a}D, \quad c_4 = b + c\bar{D}, \quad d_4 = a + b\bar{D}, \quad l_5 = a\bar{D},$$

$$a_5 = d, \quad b_5 = c + \bar{a}d, \quad c_5 = b + \bar{a}c, \quad d_5 = a + \bar{a}b, \quad l_6 = a\bar{a}.$$

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