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FLEXURAL VIBRATION IN A HEAT CONDUCTING CYLINDRICAL PANEL RESTING ON WINKLER ELASTIC FOUNDATION

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Abstract. Flexural vibration in a homogeneous isotropic heat conducting cylindrical panel resting on the elastic medium (Winkler model) is investigated in the context of Coupled theory of thermoelasticity (CT) and Lord-Shulman (LS) generalized theory of thermoelasticity. The analysis is carried out by introducing three displacement potential functions so that the equations of motion are uncoupled and simplified**.** A modified Bessel function solution with complex arguments is then directly used for the case of complex eigen values. In order to illustrate theoretical development, numerical solutions are obtained for non-dimensional frequency, attenuation coefficient (symmetric and skew symmetric) and are presented graphically for a zinc material. The numerical results indicate that the effect of thermal relaxation time and the damping of embedded medium on the non-dimensional frequency are very pronounced and also LS model is suitable for elastic material.

1. Introduction

Studies on cylindrical panel structures and the consideration of effects such as elastic foundations on their behavior are among one of the most important research fields in applied mechanics. The analysis of thermally induced flexural wave propagation of cylindrical panel embedded in a elastic medium is common place in the design of structures, atomic reactors, steam turbines, wave loading on submarine, the impact loading due to superfast train and jets and other devices operating at elevated temperature. Moreover, it is well recognized that the investigation of the thermal effects on elastic wave propagation supported by elastic foundation has bearing on many seismological application. The simplest model presented here for the elastic foundation is the Winkler model, which assumes that the shear resistance of the foundation is ignorable, compared to the shear capacity of the foundation, and models the foundation as a set of independent springs. This model enables the foundation to act more interestingly by having rigid points in its domain. When stress wave propagates along embedded structures, they are constrained between its geometric boundaries and they undergo multiple reflections. A complex mixture of constructive and destructive interferences arises from successive reflections, refractions and mode conversion due to the interaction between waves and embedding medium. So, this type of study may be used in applications involving nondestructive testing (NDT), qualitative nondestructive evaluation (QNDE) of large diameter pipes and health monitoring of other ailing infrastructures in addition to check and verify the validity of FEM and BEM for such problems.

The static analysis cannot predict the behavior of the material due to the thermal stresses changes very rapidly. Therefore in case of suddenly applied loading, thermal deformation and the role of inertia are getting more important. This thermo elastic stress response being significant leads to the propagation of thermo elastic stress waves in solids. The theory of coupling of thermal and strain fields gives rise to Coupled Thermoelasticity (CT) and was first postulated by Duhamel [1], shortly after the theory of elasticity. The theory of thermoelasticity is well established by Nowacki [2]. Lord and Shulman [3] and Green and Lindsay [4] modified the Fourier law and constitutive relations, so as to get hyperbolic equation for heat conduction by taking into account the time needed for acceleration of heat flow and relaxation of stresses. A special feature of the Green–Lindsay model is that it does not violate the classical Fourier's heat conduction law. Vibration of functionally graded multilayered orthotropic cylindrical panel under thermo mechanical load was analyzed by Wang et.al [5]. Gao and Noda [6] have studied the thermal-induced interfacial cracking of magneto electro elastic materials under uniform heat flow. Chen et al. [7] analyzed the point temperature solution for a penny-shaped crack in an infinite transversely isotropic thermo-piezo-elastic medium subjected to a concentrated thermal load applied arbitrarily at the crack surface using the generalized potential theory. Banerjee and Pao [8] investigated the propagation of plane harmonic waves in infinitely extended anisotropic solids by taking into account the thermal relaxation time. Ponnusamy [9] have obtained the frequency equation of free vibration of a generalized thermoelastic solid cylinder of arbitrary cross section by using Fourier expansion collocation method. Ponnusamy and Selvamani [10, 11] have studied respectively, the dispersion analysis of generalized magneto-thermo elastic waves in a transversely isotropic cylindrical panel and wave propagation in a generalized thermoelastic plate embedded in elastic medium using the wave propagation approach. Later, Selvamani [12] obtained mathematical modeling and analysis for damping of generalized thermoelastic waves in a homogeneous isotropic plate. Sharma [13] investigated the three dimensional vibration analysis of a transversely isotropic thermoelastic cylindrical panel. On natural frequencies of a transversely isotropic cylindrical panel on a Kerr foundation was discussed by Cai et al. [14]. Paliwal et al. [15] presented a clear investigation on the coupled free vibrations of isotropic circular cylindrical shell on Winkler and Pasternak foundations by employing a membrane theory. Wei-Ren Chen et al. [16] studied nonlinear vibration of hybrid composite plates on elastic foundations using Galerkin and Rungakutta methods. Based on a refined sinusoidal plate theory, Zenkour et al. [17] have presented an investigation on the bending response of functionally graded viscoelastic beams resting on elastic foundations. Nonlinear analysis of non-uniform beams on nonlinear elastic foundation was investigated by Tsiatas [18]. Singh et al. [19] have investigated the post buckling response of a laminated composite plate on an elastic foundation with random system properties. For structural elements on soil capable of supporting compressive reactions only, the tensionless foundation model should be adopted for realistic results Celeb et al. [20]. Recently, Zenkour et al. [21] discussed the simple and mixed first-order theories for plates resting on elastic foundations. Later, the three dimensional elasticity study of vibration of a composite shell panel with embedded piezo electric sensors was analyzed by Alireza Daneshmehr [22].

In this paper the flexural vibration in a homogeneous isotropic heat conducting cylindrical panel resting on an elastic medium (Winkler model) is investigated in the context of generalized theory of thermo elasticity. The analysis is carried out by introducing three displacement functions so that the equations of motion are uncoupled and simplified**.** A modified Bessel function solution with complex arguments is then directly used for the case of complex Eigen values. In order to illustrate theoretical development, numerical solutions are obtained and presented graphically for a zinc material.

2. The governing equations

Consider a cylindrical panel embedded on elastic medium of length L having inner and outer radius *a* and *b* with thickness *h*. The angle subtended by the cylindrical panel, which is known as center angle, is denoted by α. The deformation of the cylindrical panel in the direction r, θ, and z are defined by u, v and w. The cylindrical panel is assumed to be homogenous, isotropic and linearly elastic with Young's modulus E, Poisson ratio v and density ρ in an undisturbed state.

In cylindrical coordinate the three dimensional stress equation of motion, strain displacement relation in the absence of body force for a linearly elastic medium are

$$\sigma_{rr,r} + r^{-1}\sigma_{r\theta,\theta} + \sigma_{rz,z} + r^{-1}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) = \rho u_{,tt}, \tag{1a}$$

$$\sigma_{r\theta,r} + r^{-1}\sigma_{\theta\theta,\theta} + \sigma_{,rzz} + \sigma_{\theta z,z} + 2r^{-1}\sigma_{r\theta} = \rho v_{,tt}, \tag{1b}$$

$$\sigma_{rz,r} + r^{-1}\sigma_{\theta z,\theta} + \sigma_{zz,z} + r^{-1}\sigma_{r\theta} = \rho w_{,tt}. \tag{1c}$$

The Lord-Shulman heat conduction equation is given by

$$K\left(T_{,rr} + r^{-1}T_{,r} + r^{-2}T_{,\theta\theta} + T_{,zz}\right) - \rho C_v\left(T_{,t} + \tau_0 T_{,tt}\right) = \beta T_0\left(\frac{\partial}{\partial t} + \delta_{1k}\tau_0\frac{\partial^2}{\partial t^2}\right)\left(u_{,r} + r^{-1}\left(u + v_{,\theta}\right) + w_{,z}\right), \tag{1d}$$

where ρ is the mass density, c_v is the specific heat capacity, $\kappa = K/\rho c_v$ is the diffusity, K is the thermal conductivity, T_0 is the reference temperature.
The stress strain relation is given by generalized Hook's law

$$\sigma_{rr} = \lambda(e_{rr} + e_{\theta\theta} + e_{zz}) + 2\mu e_{rr} - \beta(T + \tau_1\delta_{2k}T_{,t}), \tag{2a}$$

$$\sigma_{\theta\theta} = \lambda(e_{rr} + e_{\theta\theta} + e_{zz}) + 2\mu e_{\theta\theta} - \beta(T + \tau_1\delta_{2k}T_{,t}), \tag{2b}$$

$$\sigma_{zz} = \lambda(e_{rr} + e_{\theta\theta} + e_{zz}) + 2\mu e_{zz} - \beta(T + \tau_1\delta_{2k}T_{,t}), \tag{2c}$$

where e_{ij} are the strain components, β is the thermal stress coefficients, T is the temperature, t is the time, λ and μ are Lame' constants, τ_0 is the thermal relaxation time and the comma notation is used for spatial derivatives. The strain e_{ij} are related to the displacements are given by

$$\sigma_{r\theta} = \mu\gamma_{r\theta}, \qquad \sigma_{rz} = \mu\gamma_{rz}, \qquad \sigma_{\theta z} = \mu\gamma_{\theta z}, \qquad e_{rr} = \frac{\partial u}{\partial r}, \qquad e_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial\theta}, \tag{3}$$

$$e_{zz} = \frac{\partial w}{\partial z}, \qquad \gamma_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial\theta}, \qquad \gamma_{rz} = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}, \quad \gamma_{z\theta} = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial\theta}, \tag{4}$$

where u, v, w are displacements along radial, circumferential and axial directions respectively. $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}$ are the normal stress components and $\sigma_{r\theta}, \sigma_{\theta z}, \sigma_{zr}$ are the shear stress components, $e_{rr}, e_{\theta\theta}, e_{zz}$ are normal strain components and $e_{r\theta}, e_{\theta z}, e_{zr}$ are shear strain components.
Substitution of Eqs. (2), (3) and (4) in Eqs. (1) gives the following three displacement equations of motion

$$\left(\lambda + 2\mu\right)\left(u_{,rr} + r^{-1}u_{,r} - r^{-2}u\right) + \mu r^{-2}u_{,\theta\theta} + \mu u_{,zz} + r^{-1}\left(\lambda + \mu\right)v_{,r\theta} - r^{-2}\left(\lambda + 3\mu\right)v_{,\theta} + \left(\lambda + \mu\right)w_{,rz}$$
$$-\beta\left(T_{,r} + \tau_1\delta_{2k}T_{,rt}\right) = \rho u_{,tt},$$

$$\mu\left(v_{,rr}+r^{-1}v_{,r}-r^{-2}v\right)+r^{-2}\left(\lambda+2\mu\right)v_{,\theta\theta}+\mu v_{,zz}+r^{-2}\left(\lambda+3\mu\right)u_{,\theta}+r^{-1}\left(\lambda+\mu\right)u_{,r\theta}+r^{-1}\left(\lambda+\mu\right)w_{,\theta z}$$

$$-\beta\left(T_{,\theta}+\tau_1\delta_{2k}T_{,\theta t}\right)=\rho v_{,tt},$$

$$\left(\lambda+2\mu\right)w_{,zz}+\mu\left(w_{,rr}+r^{-1}w_{,r}+r^{-2}w_{,\theta\theta}\right)+\left(\lambda+\mu\right)u_{,rz}+r^{-1}\left(\lambda+\mu\right)v_{,\theta z}+r^{-1}\left(\lambda+\mu\right)u_{,z}$$

$$-\beta\left(T_{,z}+\tau_1\delta_{2k}T_{,zt}\right)=\rho w_{,tt},$$

$$\rho c_v\kappa\left(T_{,rr}+r^{-1}T_{,r}+r^{-2}T_{,\theta\theta}+T_{,zz}\right)=\rho c_v\left[T_{,t}+\tau_0T_{,tt}\right]+\beta T_0\left(\frac{\partial}{\partial t}+\tau_0\delta_{1k}\frac{\partial^2}{\partial t^2}\right)\left[u_{,r}+r^{-1}(u+v_{,\theta})+w_{,z}\right]. \quad (5)$$

Here the δ_{ij} is the Kronecker delta function. In addition, $k=1$ represents LS theory and $k=2$ indicates GL theory. Since the heat conduction equation in this theory is of hyperbolic wave type, it is automatically ensures that the finite speeds of propagation for heat and elastic waves.

To solve Eq. (5), we take the displacement potential as

$$u=\frac{1}{r}\psi,_{\theta}-\phi,_{r}, \quad v=-\frac{1}{r}\phi,_{\theta}-\psi,_{\varpi}, \qquad w=-\chi,_{z}, \quad (6)$$

Using Eq. (6) in Eq. (5), we find that φ, χ, T satisfy the equations

$$((\lambda+2\mu)\nabla^2{}_1+\mu\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2})\phi-(\lambda+\mu)\frac{\partial^2\chi}{\partial z^2}^2=\beta\ (T+\tau_1\delta_{2k}T_{,t}), \quad (7a)$$

$$(\mu\nabla_1{}^2+(\lambda+2\mu)\frac{\partial^2}{\partial z^2}-\rho\frac{\partial^2}{\partial t^2})\chi-(\lambda+\mu)\nabla_1{}^2\phi=\beta(T+\tau_1\delta_{2k}T_{,t}), \quad (7b)$$

$$(\nabla_1{}^2+\frac{\partial^2}{\partial z^2}-\frac{\rho}{\mu}\frac{\partial^2}{\partial t^2})\psi=0, \quad (7c)$$

$$\nabla_1{}^2T+\frac{\partial^2T}{\partial z^2}-\frac{\rho C_V i\omega\eta_1T}{K}=\frac{\beta T_0(i\omega)\eta_2}{K}(\nabla_1{}^2\phi+\frac{\partial^2\chi}{\partial z^2}), \quad (7d)$$

where $\eta_0=1+i\omega\tau_0$, $\eta_1=1+i\omega\delta_{2k}\tau_1$, $\eta_2=1+i\omega\delta_{1k}\tau_0$.

Equation (7c) in ψ gives a purely transverse wave, which is not affected by temperature. This wave is polarized in planes perpendicular to the z-axis. Under Hook's law, the particle of the medium oscillate with simple harmonic motion about their equilibrium position. Since the major characteristic of such a solution is to expect some type of pulse distortion to occur in a system, we assume that the disturbance is time harmonic through the factor $e^{i\omega t}$.

3. Solution to the problem

The equations (7) are coupled partial differential equations of the three displacement components. To uncouple Eqs. (7), we can write three displacement functions which satisfies the simply supported boundary conditions followed by [13] as follows:

$$\psi(r,\theta,z,t)=\overline{\psi}(r)\sin(m\pi z)\cos(n\pi\theta/\alpha)e^{i\omega t},$$

$$\phi(r,\theta,z,t)=\overline{\phi}(r)\sin(m\pi z)\sin(n\pi\theta/\alpha)e^{i\omega t},$$

$$\chi(r,\theta,z,t)=\overline{\chi}(r)\sin(m\pi z)\sin(n\pi\theta/\alpha)e^{i\omega t},$$

$$T(r,\theta,z,t)=\overline{T}(r)\sin(m\pi z)\sin(n\pi\theta/\alpha)e^{i\omega t}, \tag{8}$$

where $\overline{\psi}(r)$, $\overline{\phi}(r)$, $\overline{\chi}(r)$ and $\overline{T}(r)$ are the amplitude of the wave describing the maximum displacement of the particle at position r from the equilibrium position and m is the circumferential mode and n is the axial mode. By introducing the dimensionless quantities

$$r^{'}=\frac{r}{R},\ z^{'}=\frac{z}{L},\ \overline{T}=\frac{T}{T_0},\ \delta=\frac{n\pi}{\alpha},\ t_L=\frac{m\pi R}{L},\ \overline{\lambda}=\frac{\lambda}{\mu},\ \in_4=\frac{1}{2+\overline{\lambda}},\ C_1^{\,2}=\frac{\lambda+2\mu}{\rho},\ \omega^2=\frac{\Omega^2R^2}{C^2{}_1},$$

and substituting in Eq. (8), we obtain the following system of equations

$$(\nabla_2^2+k_1^{\,2})\overline{\psi}=0, \tag{9a}$$

$$(\nabla_2^2+g_1)\overline{\phi}+g_2\overline{\chi}-g_4\overline{T}=0, \tag{9b}$$

$$(\nabla_2^2+g_3)\overline{\chi}+(1+\overline{\lambda})\nabla_2^2\overline{\phi}+(2+\overline{\lambda})g_4\overline{T}=0, \tag{9c}$$

$$(\nabla_2{}^2-t_L{}^2+\in_2\Omega^2-i\in_3)\overline{T}+i\in_1\Omega\nabla_2{}^2\overline{\phi}-i\in_1\Omega t_L{}^2\overline{\chi}=0, \tag{9d}$$

where

$$\nabla_2^2=\frac{\partial^2}{\partial r^2}\frac{1}{r}\frac{\partial}{\partial r}-\frac{\delta^2}{r^2},\quad \in_1=\frac{T_0R\beta^2}{\rho^2C_VC_1K},\quad \in_2=\frac{C_1^{\,2}}{C_VK},\quad \in_3=\frac{C_1R}{K},\quad g_1=(2+\overline{\lambda})(t_L^2-\omega^2),$$

$$g_2=\in_4(1+\overline{\lambda})t_L{}^2,\ g_3=(\omega^2-\in_4 t_L{}^2),\ g_4=\frac{\beta T_0R^2\eta_1}{\lambda+2\mu},\ g_5=\in_1\omega.$$

Nontrivial solutions of Eqs. (9) exists when the determinant of the coefficient of Eqs. (9) is equal to zero

$$\begin{vmatrix}(\nabla_2{}^2+g_1) & -g_2 & g_4\\ (1+\overline{\lambda})\nabla_2{}^2 & (\nabla_2{}^2+g_3) & (2+\overline{\lambda})g_4\\ ig_5\nabla_2{}^2 & -ig_5t_L{}^2 & (\nabla_2{}^2-t_L{}^2+\in_2\omega^2-i\omega\in_3)\end{vmatrix}(\overline{\phi},\overline{\chi},\overline{T})=0. \tag{10}$$

Equation (10), on simplification reduces to the following differential equation:

$$\nabla_2^6+A\nabla_2^4+B\nabla_2^2+C=0, \tag{11}$$

where,

$$A=-g_1+g_2(1+\overline{\lambda})+g_3-g_4g_5it_L{}^2+\in_2\omega^2-i\in_3\Omega,$$

$$B=-g_1g_3-g_1g_4g_5i-g_2g_4g_5i(2+\overline{\lambda})+t_l{}^2(g_1-g_2-g_3)+g_4g_5it_L{}^2+g_2\omega^2\in_2(1+\overline{\lambda})-g_2i\in_3\omega(1+\overline{\lambda}),$$

$$-g_2t_L{}^2\overline{\lambda}+g_3\omega(\omega\in_2-i\in_3)+g_1\omega(i\in_3-\omega\in_2),$$

$$C=g_1g_3(t_L{}^2+i\in_3\omega-\in_2\omega^2)+ig_3g_4g_5t_L{}^2(2+\overline{\lambda}).$$

The solutions of Eq. (11) for flexural mode are

$$\overline{\phi}(r)=\sum_{i=1}^{3}\left[A_iI_\delta(\alpha_ir)+B_iK_\delta(\alpha_ir)\right]\mathrm{co}\ \ n\theta\mathrm{s},$$

$$\overline{\chi}(r)=\sum_{i=1}^{3} d_i\left[A_i I_\delta(\alpha_i r)+B_i K_\delta(\alpha_i r)\right]\cos n\theta,$$

$$\overline{T}(r)=\sum_{i=1}^{3} e_i\left[A_i I_\delta(\alpha_i r)+B_i K_\delta(\alpha_i r)\right]\cos n\theta. \tag{12}$$

Here, $(\alpha_i r)^2 \quad i=1,2,3$ are the non-zero roots of the algebraic equation

$$(\alpha_i r)^6 - A(\alpha_i r)^4 + B(\alpha_i r)^2 - C = 0.$$

The arbitrary constant d_i and e_i is obtained from

$$d_i=\left[\frac{\left(1+\overline{\lambda}\right)\delta_i^2-(2+\overline{\lambda})\delta_i^2-g_1}{g_2(2+\overline{\lambda})-\delta_i^2-g_3}\right],$$

$$e_i=\left(\frac{\lambda+2\overline{\mu}}{\beta T_0 R^2}\right)\left[\frac{\varepsilon_4\delta_i^2+\left(\varepsilon_4\left(g_1+g_3\right)+\varepsilon_4(1+\overline{\lambda})g_2\right)\delta_i^2+\varepsilon_4 g_1 g_3+\delta_i^2)-g_1 g_3}{\varepsilon_4 g_3+\varepsilon_4\delta_i^2-g_2}\right]. \tag{13}$$

Equation (9a) is a Bessel equations with its possible solutions are

$$\overline{\psi}=\begin{cases} A_4 J_\delta(k_1 r)+B_4 Y_\delta(k_1 r), & k_1^2>0 \\ A_4 r^\delta + B_4 r^{-\delta}, & k_1^2=0 \\ A_4 I_\delta(k_1' r)+B_4 K_\delta(k_1' r), & k_1^2<0 \end{cases}, \tag{14}$$

where $k_1'^2=-k_1^2$ and J_δ and Y_δ are Bessel functions of the first and second kinds respectively while, I_δ and k_δ are modified Bessel functions of first and second kinds respectively. A_i, B_i $i=1,2,3,4$ are the arbitrary constants. Generally, $k_1^2\neq 0$, so that the situation $k_1^2\neq 0$ will not be discussed in the following. For convenience, we consider the case of $k_1^2>0$ and the derivation for the case of $k_1^2<0$ is similar.

The solution of Eq. (9a) is

$$\overline{\psi}(r)=A_4 I_\delta(k_1 r)+B_4 K_\delta(k_1 r) \quad , \tag{15}$$

where $k_1^2=(2+\overline{\lambda})\omega^2-t_L^2$.

3.1. Displacement, temperature and stress changes. In this section we shall derive the secular equation for the three dimensional vibrations cylindrical panel subjected to traction free boundary conditions at the upper and lower surfaces at $r=a,b$:

$$u=\left(-\overline{\phi}'-\frac{\delta\overline{\psi}'}{r}\right)\sin(m\pi z)\sin(\delta\theta)e^{i\omega t},$$

$$v=\left(-\overline{\psi}'-\frac{\delta\overline{\phi}'}{r}\right)\sin(m\pi z)\cos(\delta\theta)e^{i\omega t},$$

$$w=\overline{\chi}t_L\cos(m\pi z)\sin(\delta\theta)e^{i\omega t} \quad ,$$

$$T=\overline{T}\sin(m\pi z)\cos(\delta\theta)e^{i\omega t},$$

$$\overline{\sigma}_{rr}=$$

$$\left[\left(2+\overline{\lambda}\right)\delta\left(\frac{\overline{\psi}'}{r}-\frac{\overline{\psi}}{r^2}\right)+\left(2+\overline{\lambda}\right)\left(\frac{1}{r}\overline{\varphi}'+(\alpha_i^2-\frac{\delta^2}{r^2}\overline{\varphi})\right)+\overline{\lambda}\left(\frac{\delta}{r^2}\overline{\psi}-\frac{1}{r}\overline{\phi}'-\frac{\delta^2}{r^2}\overline{\varphi}-\frac{\delta}{r}\overline{\psi}'-t_L^2\overline{\chi}\right)\right]\sin(m\pi)z\cos(\delta\theta)e^{i\omega t},$$

$$\overline{\sigma}_{r\theta}=2\left(\frac{1}{r}\overline{\psi}+(\alpha_i^2-\frac{\delta^2}{r^2})\overline{\psi}-\frac{2\delta}{r}\overline{\varphi}'+\frac{2\delta}{r^2}\overline{\varphi}+\frac{\overline{\psi}'}{r}-\frac{\delta^2}{r^2}\overline{\psi}\right)\sin(m\pi)z\cos(\delta\theta)e^{i\omega t},$$

$$\overline{\sigma}_{rz}=2t_L\left(-\overline{\varphi}'-\frac{\delta}{r}\overline{\psi}+\overline{\chi}'\right)\cos(m\pi)z\sin(\delta\theta)e^{i\omega t}, \tag{16}$$

where prime denotes the differentiation with respect to r, $\overline{u}_i=u_i/R$, $(i=r,\theta,z)$ are three non-dimensional displacements and $\overline{\sigma}_{rr}=\sigma_{rr}/\mu$, $\overline{\sigma}_{r\theta}=\sigma_{r\theta}/\mu$, $\overline{\sigma}_{rz}=\sigma_{rz}/\mu$ are three non-dimensional stresses.

4. Boundary conditions

In this section we shall derive the secular equation for the three dimensional vibrations cylindrical panel subjected to traction free boundary conditions at the upper and lower surfaces at $r=a,b$.

(i) The traction free non dimensional mechanical boundary conditions for a stress free edge are given by

$$\sigma_{rr}=\sigma_{r\theta}=\sigma_{rz}=0, \tag{17a}$$

(ii) The non dimensional insulated or isothermal thermal boundary condition is given by

$$T_{,r}+hT=0, \tag{17b}$$

where h is the surface heat transfer coefficient. Here $h\to 0$ corresponds to thermally insulated surface and $h\to\infty$ refers to isothermal one.

Since two-parameter models naturally require both parameters for practical applications, methods for determining the parameters have been discussed for the past several decades. Pasternak himself suggested using plate loading tests for two parameter evaluations, but he didn't give the parameters actual values. If the shear modulus G goes to zero, the Pasternak model will degenerate in to the Winkler modeling; one of the most fundamental methods was suggested by Winkler. The approach introduces a linear algebraic relationship between the normal displacement of the structure and the contact pressure. Structures which are supported along their length such as beams or pipelines resting on elastic soil are very commonly modeled with a Winkler foundation. The model originated from "Winkler's hypothesis," which states that the deflection at any point on the surface of an elastic continuum is proportional only to the load being applied to the surface and is independent of the load applied to any other points on the surface [23].

5. Frequency equations

For the purpose of comparison, we first consider the uncoupled free vibration of isotropic cylindrical panel. In this case both convex and concave surface of the panel are traction free

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{rz} = 0, \qquad T_{,r} = 0, \ \ (r = a, b)\,, \tag{18}$$

$$\left|E_{ij}^{1}\right| = 0\,, \qquad i,\ j = 1, 2 \,.....\, 8\,. \tag{19}$$

The constants in the determinant are obtained as follows:

$$E_{11}^{1} = (2+\overline{\lambda})\left((\delta I_{\delta}(\alpha_1 t_1)/t_1^{\,2} - \frac{\alpha_1}{t_1} I_{\delta+1}(\alpha_1 t_1)) - ((\alpha_1 t_1)^2 R^2 - \delta^2) I_{\delta}(\alpha_1 t_1)/t_1^{\,2} \right),$$

$$+\overline{\lambda}\left(\delta(\delta-1) I_{\delta}(\alpha_1 t_1)/t_1^{\,2} - \frac{\alpha_1}{t_1} I_{\delta+1}(\alpha_1 t_1) \right) + \overline{\lambda} d_1 t_L^{\,2} I_{\delta}(\alpha_1 t_1) - \beta T_0 R^2 e_1 \overline{\lambda}\,,$$

$$E_{13}^{1} = (2+\overline{\lambda})\left((\delta I_{\delta}(\alpha_2 t_1)/t_1^{\,2} - \frac{\alpha_2}{t_2} I_{\delta+1}(\alpha_2 t_1)) - ((\alpha_2 t_1)^2 R^2 - \delta^2) I_{\delta}(\alpha_2 t_1)/t_1^{\,2} \right),$$

$$+\overline{\lambda}\left(\delta(\delta-1) I_{\delta}(\alpha_2 t_1)/t_1^{\,2} - \frac{\alpha_2}{t_1} I_{\delta+1}(\alpha_2 t_1) \right) + \overline{\lambda} d_2 t_L^{\,2} I_{\delta}(\alpha_2 t_1) - \beta T_0 R^2 e_2 \overline{\lambda}\,,$$

$$E_{15}^{1} = (2+\overline{\lambda})\left((\delta I_{\delta}(\alpha_3 t_1)/t_1^{\,2} - \frac{\alpha_2}{t_2} I_{\delta+1}(\alpha_3 t_1)) - ((\alpha_3 t_1)^2 R^2 - \delta^2) I_{\delta}(\alpha_3 t_1)/t_1^{\,2} \right),$$

$$+\overline{\lambda}\left(\delta(\delta-1) I_{\delta}(\alpha_3 t_1)/t_1^{\,2} - \frac{\alpha_2}{t_1} I_{\delta+1}(\alpha_3 t_1) \right) + \overline{\lambda} d_3 t_L^{\,2} I_{\delta}(\alpha_3 t_1) - \beta T_0 R^2 e_3 \overline{\lambda}\,,$$

$$E_{17}^{1} = (2+\overline{\lambda})\left((\frac{k_1\delta}{t_1} I_{\delta+1}(k_1 t_1) - \delta(\delta-1) I_{\delta}(k_1 t_1)/t_1^{\,2} \right) + \overline{\lambda}\left(\delta(\delta-1) I_{\delta}(k_1 t_1)/t_1^{\,2} - \frac{k_1\delta}{t_1} I_{\delta+1}(k_1 t_1) \right),$$

$$E_{21}^{1} = 2\delta\big((\alpha_1/t_1) I_{\delta+1}(\alpha_1 t_1) - \delta(\delta-1) I_{\delta}(\alpha_1 t_1) \big),$$

$$E_{23}^{1} = 2\delta\big((\alpha_2/t_1) I_{\delta+1}(\alpha_2 t_1) - \delta(\delta-1) I_{\delta}(\alpha_2 t_1) \big),$$

$$E_{25}^{1} = 2\delta\big((\alpha_3/t_1) I_{\delta+1}(\alpha_3 t_1) - \delta(\delta-1) I_{\delta}(\alpha_3 t_1) \big),$$

$$E_{27}^{1} = (k_1 t_1)^2 R^2 I_{\delta}(k_1 t_1) - 2\delta(\delta-1) I_{\delta}(k_1 t_1)/t_1^{\,2} + k_1/t_1 I_{\delta+1}(k_1 t_1)\,,$$

$$E_{31}^{1} = -t_L(1+d_1)\big(\delta/t_1 I_{\delta}(\alpha_1 t_1) - \alpha_1 I_{\delta+1}(\alpha_1 t_1) \big),$$

$$E_{33}^{1} = -t_L(1+d_2)\big(\delta/t_1 I_{\delta}(\alpha_2 t_1) - \alpha_2 I_{\delta+1}(\alpha_2 t_1) \big),$$

$$E_{35}^{1} = -t_L(1+d_3)\big(\delta/t_1 I_{\delta}(\alpha_3 t_1) - \alpha_2 I_{\delta+1}(\alpha_3 t_1) \big)$$

$$E_{37}^{1} = -t_L(\delta/t_1) I_{\delta}(k_1 t_1)\,,$$

$$E_{41}^{1} = e_1[(\delta/t_1) I_{\delta}(\alpha_1 t_1) - (\alpha_1) I_{\delta+1}(\alpha_1 t_1)],$$

$$E_{43}^{1} = e_2[(\delta/t_1) I_{\delta}(\alpha_2 t_1) - (\alpha_2) I_{\delta+1}(\alpha_2 t_1)]\,,$$

$$E_{4\,5}^{1}= e_3[(\delta/t_1)I_{\delta}(\alpha_3 t_1)-(\alpha_3)I_{\delta+1}(\alpha_3 t_1)],$$

$$E_{47}^{1}=0,$$

in which $t_1 = a/R = 1-t^*/2$, $t_2 = b/R = 1+t^*/2$ and $t^* = b-a/R$ is the thickness -to-mean radius ratio of the panel. Obviously $E_{ij}\,(j=2,4,6,8)$ can obtained by just replacing modified Bessel function of the first kind in $E_{ij}\,(i=1,3,5,7)$ with the ones of the second kind, respectively, while $E_{ij}\,(i=5,6,7,8)$ can be obtained by just replacing t_1 in $E_{ij}\,(i=1,2,3,4)$ with t_2. Now we consider the coupled free vibration problem. Allowing for the effect of the surrounded elastic medium, which is treated as the Pasternak model, the boundary conditions at the inner and outer surfaces $r = a, b$ can consider as follows

$$\sigma_{rr}=\sigma_{r\theta}=\sigma_{rz}=0, \qquad T_{,r}=0, \quad (r=a), \tag{20}$$

$$\sigma_{rr}=-\kappa u+G\Delta u, \qquad \sigma_{r\theta}=\sigma_{rz}=0, \qquad (r=b), \tag{21}$$

where $\Delta=\partial^2/\partial z^2+\left(1/r^2\right)\partial^2/\partial\theta^2$, κ is the foundation modulus and G is the shear modulus of the foundation. It is mentioned here that the elastic medium can be modeled as Winkler type Cai et al. [14] by setting G=0 in Eq. (21). From Eq. (21), and the results obtained in the preceding section, we get the coupled free vibration frequency equation as follows:

$$\left|E_{ij}^{2}\right|=0, \qquad i,j=1,2.....8, \tag{22}$$

$$E_{ij}^{2}=E_{ij}^{1}, \qquad \left(i=1,2,3,4,6,7,8; \quad j=1,2....8\right),$$

$$E_{51}^{2}=E_{51}^{1}-p\delta I_{\delta}\left(\alpha_1 t_2\right)/t_2, \qquad E_{52}^{2}=E_{52}^{1}-p\delta K_{\delta}\left(\alpha_1 t_2\right)/t_2,$$

$$E_{53}^{2}=E_{53}^{1}-p\left(\delta/t_2 I_{\delta}\left(\alpha_2 t_2\right)-t_2 I_{\delta+1}\left(\alpha_2 t_2\right)\right), \quad E_{54}^{2}=E_{54}^{1}-p\left(\delta/t_2 K_{\delta}\left(\alpha_2 t_2\right)-t_2 K_{\delta+1}\left(\alpha_2 t_2\right)\right),$$

$$E_{55}^{2}=E_{55}^{1}-p\left(\delta/t_2 I_{\delta}\left(\alpha_3 t_2\right)-t_2 I_{\delta+1}\left(\alpha_3 t_2\right)\right), \quad E_{56}^{2}=E_{56}^{1}-p\left(\delta/t_2 K_{\delta}\left(\alpha_3 t_2\right)-t_2 K_{\delta+1}\left(\alpha_3 t_2\right)\right),$$

$$E_{57}^{2}=E_{57}^{1}-p\left(\delta/t_2 I_{\delta}\left(k_1 t_2\right)-t_2 I_{\delta+1}\left(k_1 t_2\right)\right), \quad E_{58}^{2}=E_{58}^{1}-p\left(\delta/t_2 K_{\delta}\left(k_1 t_2\right)-t_2 K_{\delta+1}\left(k_1 t_2\right)\right),$$

where

$$p=p_1+p_2\left(t_L^2+n^2/t_2^2\right), \quad p_1=\kappa R/\mu \text{ and } \quad p_2=G/R\mu.$$

6. Numerical results and discussion

The coupled free flexural vibration in a simply supported homogenous isotropic thermo elastic cylindrical panel resting in a Winkler type of elastic medium is numerically solved for Zinc material by setting $p_2 = 0$. The Winkler elastic modulus $\kappa = 1.5\times10^7$. For the purpose of numerical computation we consider the closed circular cylindrical shell with the center angle $\alpha = 2\pi$ and the integer *n* must be even since the shell vibrates in circumferential full wave. The frequency equation for a closed cylindrical shell can be obtained by setting $\delta = l\,(l=1,2,3.....)$, where l is the circumferential wave number in Eq. (19). The material properties of a Zinc is

$\rho = 7.14\times10^3 kg\ m^{-3}$, $T_0 = 296\ K$, $K = 1.24\times10^2 W\ m^{-1}\deg^{-1}$,

$\mu = 0.508\times10^{11} N\ m^{-2}$, $\beta = 5.75\times10^6 N\ m^{-2}\deg^{-1}$, $\in_1 = 0.0221$,

$\lambda = 0.385\times10^{11} N\ m^{-2}$, and $C_v = 3.9\times10^2\ J\ kg^{-1}\deg^{-1}$.

The roots of the algebraic Eq. (11) were calculated using a combination of Birge-Vita method and Newton-Raphson method. In the present case simple Birge-Vita method does not work for finding the root of the algebraic equation. After obtaining the roots of the algebraic equation using Birge-Vita method, the roots are corrected for the desired accuracy using the Newton-Raphson method. This combination has overcome the difficulties in finding the roots of the algebraic equations of the governing equations. Here the values of the thermal relaxation times are calculated as $t_0 = 0.75\times10^{-13}$ sec and $t_1 = 0.5\times10^{-13}$ sec. Because the algebraic Eq. (11) contains all the information about wave speed and angular frequency and the roots are complex for all consider values of wave number, therefore the waves are attenuated in space. Then, we can write

$c^{-1} = v^{-1} + i\omega^{-1}q$,

so that $\delta = R + iq$, where $R = \omega/v$ and the wave speed (v) and the attenuation coefficient (q) are real numbers. A comparison is made for the non dimensional frequencies of flexural modes of vibration among the Lord-Shulman Theory (LS) and Classical Theory (CT) of thermo-elasticity for different thickness to mean radius ratio (t^*) parameter of a cylindrical panel with thermally insulated and isothermal boundary conditions in Table 1 and Table 2, respectively. From Table 1 and Table 2 it is clear that as the thickness parameter increases, the non dimensional frequencies are also increases in both LS and CT cases. From the above table values it is clear that the non dimensional frequency profiles exhibits high amplitude for LS theory compared with CT due to the combine effect of thermal relaxation times and damping of surrounding medium. The numerical results indicate that the effect of thermal relaxation times and the damping of embedded medium on the non dimensional frequency are very pronounced and also LS model is suitable for elastic material.

Table 1. Comparison of non-dimensional frequencies of flexural modes of vibration between the Lord-Shulman (LS) and Coupled Theory (CT) of thermoelasticities for thermally insulated cylindrical panel ($\alpha = 120$, $t_L = 0.4$, $p_1 = 0.002$, $p_2 = 0$).

t^*	LS						CT				
	n=1		n=2		n=3		n=1		n=2		n=3
0.01	0.2673		0.0565		0.0239		0.1708		0.1342		0.1152
0.05	0.3435		0.2719		0.0441		0.3255		0.1969		0.1564
0.1	0.6337		0.3977		0.2174		0.5773		0.3248		0.2544
0.3	0.8292		0.4385		0.2994		0.6941		0.5593		0.3487
0.5	1.2408		0.7052		0.3964		0.7303		0.8050		0.6584
0.7	1.4579		0.8714		0.4051		0.8070		0.8512		0.8551
0.8	1.7707		1.1350		0.8478		1.2007		1.0230		1.0138

Table 2. Comparison of non-dimensional frequencies of flexural modes of vibration between the Lord-Shulman (LS) and Coupled Theory (CT) of thermoelasticities for isothermal cylindrical panel. ($\alpha = 120$, $t_L = 0.4$, $p_1 = 0.002$, $p_2 = 0$).

t^*	LS						CT				
	n=1		n=2		n=3		n=1		n=2		n=3
0.01	0.2072		0.0465		0.0159		0.1508		0.1252		0.0132
0.05	0.4335		0.0719		0.0541		0.2255		0.1769		0.1064
0.1	0.5337		0.4977		0.1174		0.5773		0.3248		0.2444
0.3	0.8292		0.4385		0.1994		0.5941		0.4593		0.4487
0.5	0.1408		0.6952		0.2964		0.6303		0.7050		0.5584
0.7	1.6579		0.8714		0.4051		0.7070		0.8512		0.7051
0.8	1.7007		1.0550		0.6478		1.1907		1.0130		0.9038

6.1. Dispersion curves. In Figures 1 and 2 the dispersion of frequency with wave number is studied for both C T and L S theory of thermo elasticity for the symmetric and skew symmetric modes of vibration for thermally insulated cylindrical shell. From the Figure 1 the frequency increases exponentially with increasing wave number for symmetric mode of vibration. But there is dispersion in the frequency in the current range of wave number in Figure 2 for skew symmetric mode due to the combine effect of damping and insulation. On comparison on both symmetric and skew symmetric cases the LS theory attains maximum value due the relaxation times.

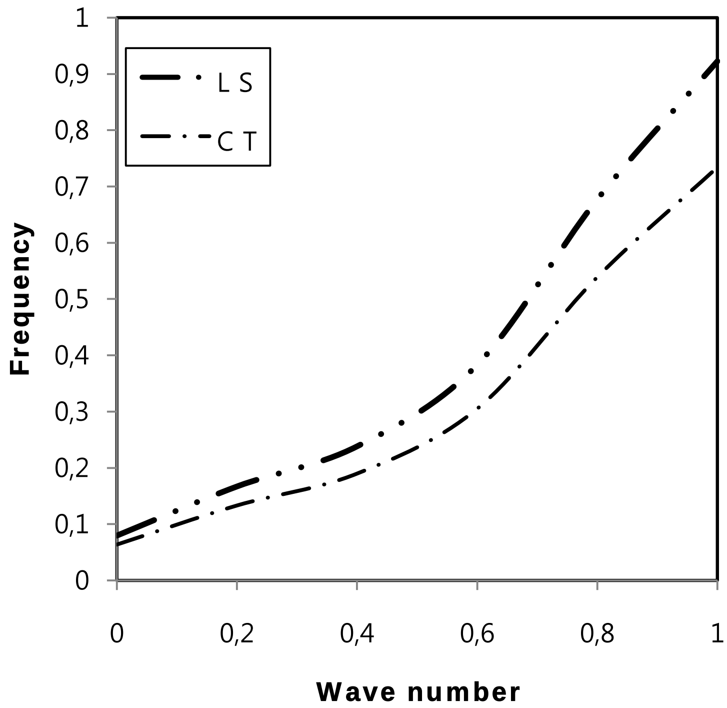

Fig. 1. Variation of frequency of cylindrical shell with wave number for symmetric mode on elastic foundation $\left(\nu = 0.3, \quad \kappa = 1.5\times10^{7}, \quad p_2 = 0\right)$.

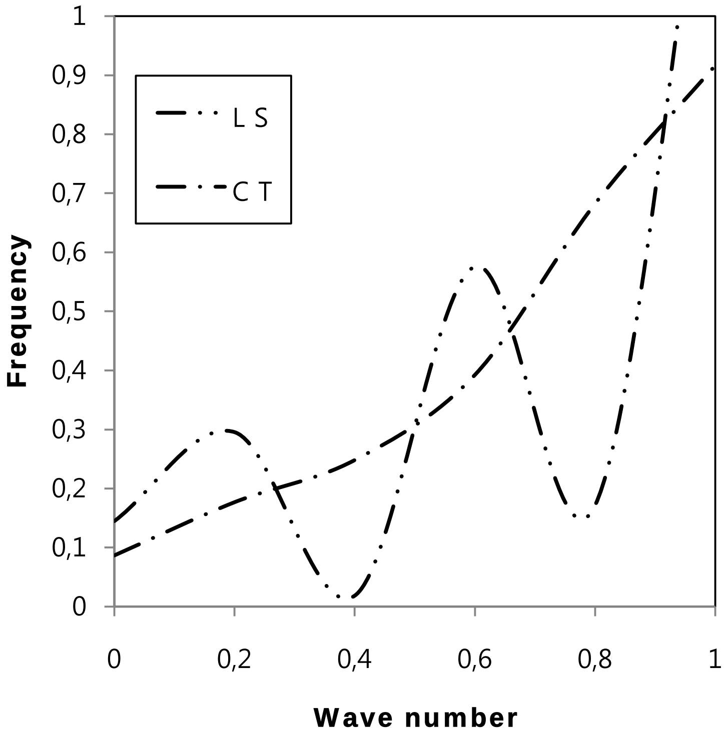

Fig. 2. Variation of attenuation coefficient of cylindrical shell with wave number for skew symmetric mode on elastic foundation $\left(\nu = 0.3, \quad \kappa = 1.5\times10^{7}, \quad p_2 = 0\right)$.

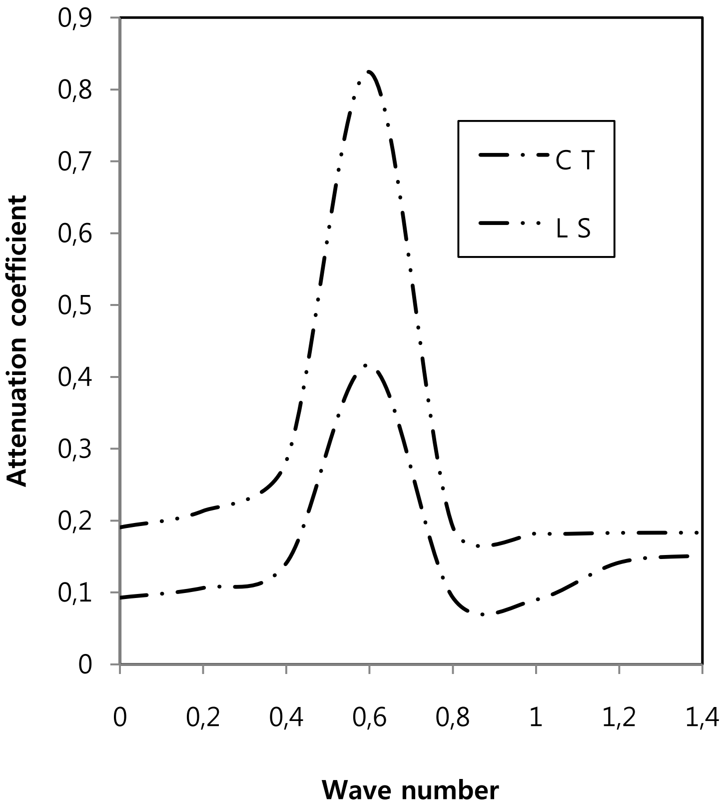

Fig. 3. Variation of attenuation coefficient of cylindrical shell with wave number for symmetric mode on elastic foundation $\left(\nu = 0.3, \quad \kappa = 1.5\times10^{7}, \quad p_2 = 0\right)$.

In Figure 3 the variation of attenuation coefficient with respect to wave number of cylindrical shell is discussed for CT and LS in symmetric mode. The magnitude of the attenuation coefficient increases monotonically to attain maximum value in $0.4 \le \delta \le 0.8$ for both CT and LS and slashes down to became asymptotically linear in the remaining range of circumferential wave number. The variation of attenuation coefficient with respect to wave number of cylindrical shell is discussed for skew symmetric mode in Fig. 4, here the attenuation coefficient attains maximum value in $0 \le \delta \le 0.4$ for both CT and LS theories and slashes down to become linear due relaxation times.

From Figure 3 and Figure 4, it is clear that the attenuation profiles exhibits high amplitude for LS theory compared with CT due to the combine effect of thermal relaxation times and damping effect of foundation. Also, from these figures it is observed that the thermal relaxation time has effect on low-wave number in the limiting case which supports the conclusion that the "second sound" effects are short lived.

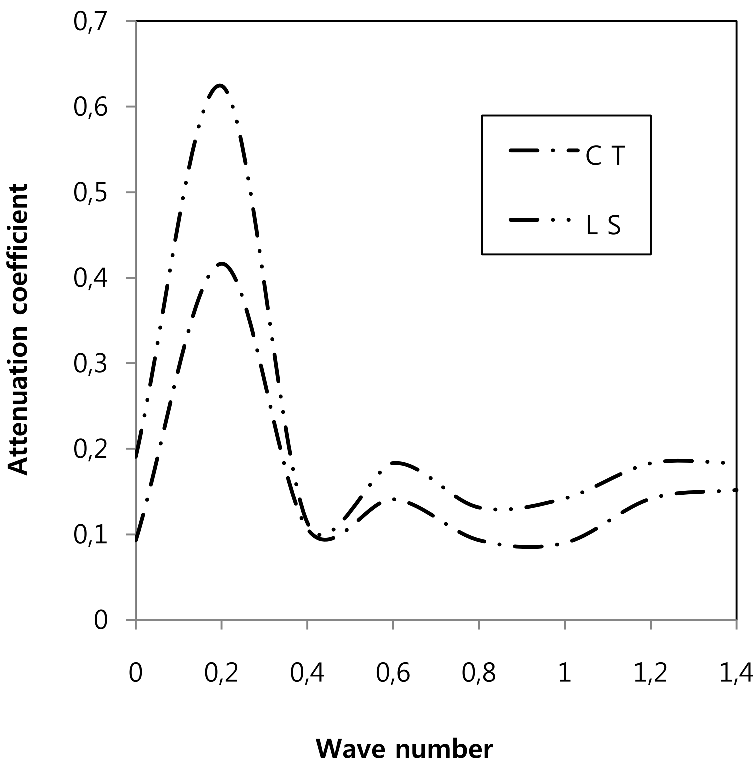

Fig. 4. Variation of attenuation coefficient of cylindrical shell with wave number for skew symmetric mode on elastic foundation $\left(\nu = 0.3, \quad \kappa = 1.5\times10^{7}, \quad p_2 = 0\right)$.

7. Conclusion

The three dimensional flexural vibration in a homogeneous isotropic thermo elastic cylindrical panel resting on the Winkler type of elastic foundation has been considered for this paper. For this problem, the governing equations of three dimensional linear theory of generalized thermoelasticity have been employed in the context of Lord-Shulman theory and solved by modified Bessel function with complex argument. The effect of the frequency and attenuation coefficient against wave number of a closed Zinc cylindrical shell is investigated and the results are presented as dispersion curves. In addition, a comparative study is made between LS and CT theories and the frequency change is observed to be high in LS followed by CT theory due to the thermal relaxation effects and damping of surrounding medium.

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