Materials Physics and Mechanics 17 (2013) 164-177 Received: September 11, 2013

ELASTO DYNAMIC WAVE PROPAGATION IN A TRANSVERSELY ISOTROPIC PIEZOELECTRIC CIRCULAR PLATE IMMERSED IN FLUID

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Abstract. The elasto dynamic wave propagation in a piezoelectric plate immersed in fluid is studied based on the three dimensional theory of linear elasticity. Three displacement potential functions are used to uncouple the equations of motion in radial, circumferential and axial directions. The frequency equations that include the interaction between the plate and fluid are obtained by the perfect-slip boundary conditions using the Bessel function solutions. The numerical calculations are carried out for the material PZT-4 and the computed non-dimensional frequency, phase velocity and attenuation coefficient are plotted as the dispersion curves for the immersed plate with different velocity ratio.

1. Introduction

To construct the sensors and transducers piezoelectric materials have been used extensively due to their direct and converse piezoelectricity effects. The direct piezoelectric effect is used in sensing applications, such as in force or displacement sensors. The converse piezoelectric effects are used in transduction applications, such as in motors and device that precisely control positioning, and in generating sonic and ultra sonic signals. In recent years, polymers piezoelectric materials have been used in numerous fields taking advantage of the flexible characteristics of these polymers. Some of the applications of these polymers include Audio device-microphones, high frequency speakers, tone generators and acoustic modems; Pressure switches – position switches, accelerometers, impact detectors, flow meters and load cells; Actuators- electronic fans and high shutters. Since piezoelectric polymers allow their use in a multitude of compositions and geometrical shapes for a large variety of applications from transducers in acoustics, ultrasonic's and hydrophone applications to resonators in band pass filters, power supplies, delay lines, medical scans and some industrial non-destructive testing instruments.

Many studies have been devoted to transverse vibration, and the natural frequencies and mode shapes for transverse vibration have been well documented [1]. Studies by Tiersten [2] should be mentioned among the early notable contributions to the topic of the mechanics of piezoelectric solids. Rajapakse and Zhou [3] solved the coupled electroelastic equations for a long piezoceramic cylinder by applying Fourier integral transforms. Berg et al. [4] assumed electric field not to be constant over the thickness of piezoceramic cylindrical shells. Ebenezer and Ramesh [5] analyzed axially polarized piezoelectric cylinders with arbitrary boundary conditions on the flat surfaces using the Bessel series. Paper by Wang [6] should be mentioned among the studies of cylindrical shells with a piezoelectric coat. Shul'ga [7]

$$\{e\} = [e_{rr}, e_{\theta\theta}, e_{r\theta}, e_{\theta z}, e_{rz}]^T,$$

$$\{D\} = [D_r, D_\theta, D_z]^T, \tag{2}$$

where $[C]$, $[\eta]$ and $[\varepsilon]$ denotes the matrices of elastic constants, piezoelectric constants and dielectric constants respectively. The matrices $[C]$, $[\eta]$ and $[\varepsilon]$ for the transversely isotropic material is given by

$$[C] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix},$$

$$[\eta] = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{bmatrix}, \quad [\varepsilon] = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix}. \tag{3}$$

The elastic, the piezoelectric, and dielectric matrices of the 6mm crystal class, the piezoelectric relations are

$$\sigma_{rr} = c_{11}e_{rr} + c_{12}e_{\theta\theta} - e_{31}E_z, \quad \sigma_{\theta\theta} = c_{12}e_{rr} + c_{11}e_{\theta\theta} - e_{31}E_z, \quad \sigma_{zz} = c_{13}e_{rr} + c_{13}e_{\theta\theta} - e_{33}E_z,$$

$$\sigma_{r\theta} = c_{66}e_{r\theta}, \quad \sigma_{rz} = 2c_{44}e_{rz} - e_{15}E_r, \quad \sigma_{\theta z} = c_{44}e_{\theta z} - e_{15}E_\theta,$$

$$D_r = e_{15}e_{rz} + \varepsilon_{11}E_r, \quad D_\theta = e_{15}e_{\theta z} + \varepsilon_{11}E_\theta \quad D_z = e_{31}(e_{rr} + e_{\theta\theta}) + \varepsilon_{33}E_z, \tag{4}$$

where $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{r\theta}, \sigma_{\theta z}, \sigma_{rz}$ are the stress components, $e_{rr}, e_{\theta\theta}, e_{zz}, e_{r\theta}, e_{\theta z}, e_{rz}$ are the strain components, $c_{11}, c_{12}, c_{13}, c_{33}, c_{44}$ and $c_{66} = (c_{11} - c_{12})/2$ are the five elastic constants, e_{31}, e_{15}, e_{33} are the piezoelectric constants, $\varepsilon_{11}, \varepsilon_{33}$ are the dielectric constants, ρ is the mass density. The comma in the subscripts denotes the partial differentiation with respect to the variables.

The strain e_{ij} are related to the displacements are given by

$$e_{rr} = u_{r,r}, \; e_{\theta\theta} = r^{-1}(u_r + u_{\theta,\theta}) \; e_{zz} = u_{z,z},$$

$$e_{r\theta} = u_{\theta,r} + r^{-1}(u_{r,\theta} - u_\theta), \quad e_{z\theta} = (u_{\theta,z} + r^{-1}u_{z,\theta}), \quad e_{rz} = u_{z,r} + u_{r,z}. \tag{5}$$

The comma in the subscripts denotes the partial differentiation with respect to the variables.

The two dimensional stress equations of motion in the absence of body force for a linearly elastic plate are obtained by the in-plane vibration as

$$\sigma_{rr,r} + r^{-1}\sigma_{r\theta,\theta} + \sigma_{rz,z} + r^{-1}(\sigma_{rr} - \sigma_{\theta\theta}) = \rho u_{r,tt},$$

$$\sigma_{r\theta,r} + r^{-1}\sigma_{\theta\theta,\theta} + \sigma_{\theta z,z} + 2r^{-1}\sigma_{r\theta} = \rho u_{\theta,tt},$$

$$V(r,\theta,z,t)=iV\exp\{i(kz+\omega t)\},$$

$$E_r(r,\theta,z,t)=-E_{,r}\exp\{i(kz+\omega t)\},$$

$$E_\theta(r,\theta,z,t)=-r^{-1}E_{,\theta}\exp\{i(kz+\omega t)\},$$

$$E_z(r,\theta,z,t)=E_{,z}\exp\{i(kz+\omega t)\},\tag{10}$$

where $i=\sqrt{-1}$, k is the wave number, ω is the angular frequency, $\phi(r,\theta),W(r,\theta)$, $\psi(r,\theta)$ and $E(r,\theta)$ are the displacement potentials and $V(r,\theta)$ is the electric potentials and a is the geometrical parameter of the plate.

By introducing the dimensionless quantities such as $x=r/a$, $\zeta=ka$, $\Omega^2=\rho\omega^2a^2/c_1^2$, $\bar{c}_{11}=c_{11}/c_{44}$, $\bar{c}_{13}=c_{13}/c_{44}$, $\bar{c}_{33}=c_{33}/c_{44}$, $\bar{c}_{66}=c_{66}/c_{44}$, $c_1^2=c_{44}/\rho$, $T_a=\sqrt{c_{11}/\rho}/a$, $\bar{z}=z/a$ and substituting Eq.(7) in Eq.(6), we obtain

$$\left(\bar{c}_{11}\nabla^2+\left(\Omega^2-\zeta^2\right)\right)\phi-\zeta\left(1+\bar{c}_{13}\right)W-\zeta\left(\bar{e}_{31}+\bar{e}_{15}\right)V=0,$$

$$\zeta\left(1+\bar{c}_{13}\right)\nabla^2\phi+\left(\nabla^2+\left(\Omega^2-\zeta^2\bar{c}_{33}\right)\right)W+\left(\bar{e}_{15}\nabla^2-\zeta^2\right)V=0,\tag{11}$$

$$\zeta\left(\bar{e}_{31}+\bar{e}_{15}\right)\nabla^2\phi+\left(\bar{e}_{15}\nabla^2-\zeta^2\right)W+\left(\zeta^2\bar{\varepsilon}_{33}-\bar{\varepsilon}_{11}\nabla^2\right)V=0,$$

and

$$\left(\bar{c}_{66}\nabla^2+\left(\Omega^2-\zeta^2\right)\right)\psi=0,\tag{12}$$

where $\nabla^2=\dfrac{\partial^2}{\partial x^2}+x^{-1}\dfrac{\partial}{\partial x}+x^{-2}\dfrac{\partial^2}{\partial\theta^2}$.

The Eq. (11) can be written as

$$\begin{vmatrix}\left(\bar{c}_{11}\nabla^2+\left(\Omega^2-\zeta^2\right)\right) & -\zeta\left(1+\bar{c}_{13}\right) & -\zeta\left(\bar{e}_{31}+\bar{e}_{15}\right)\\ \zeta\left(1+\bar{c}_{13}\right)\nabla^2 & \left(\nabla^2+\left(\Omega^2-\zeta^2\bar{c}_{33}\right)\right) & \left(\bar{e}_{15}\nabla^2-\zeta^2\right)\\ \zeta\left(\bar{e}_{31}+\bar{e}_{15}\right)\nabla^2 & \left(\bar{e}_{15}\nabla^2-\zeta^2\right) & \left(\zeta^2\bar{\varepsilon}_{33}-\bar{\varepsilon}_{11}\nabla^2\right)\end{vmatrix}\left(\phi,W,V\right)=0.\tag{13}$$

Evaluating the determinant given in Eq.(13), we obtain a partial differential equation of the form

$$\left(A\nabla^6+B\nabla^4+C\nabla^2+D\right)\left(\phi,W,V\right)=0,\tag{14}$$

where

$$A=c_{11}\left(\bar{e}_{15}^{2}+\varepsilon_{11}\right),$$

$$B=\left[\left(1+\bar{c}_{11}\right)\bar{\varepsilon}_{11}+\bar{e}_{15}^{2}\right]\Omega^2+\left\{2\left(\bar{e}_{31}+\bar{e}_{15}\right)\bar{c}_{13}\bar{e}_{15}-\left(1+\bar{\varepsilon}_{11}\bar{c}_{33}\right)\bar{c}_{11}+\bar{c}_{13}^{2}\bar{\varepsilon}_{11}+2\bar{c}_{13}\bar{\varepsilon}_{11}-2\bar{e}_{15}\bar{c}_{11}+2\bar{e}_{13}^{2}\right\}\varsigma^2,$$

$$c_f^{-2} u^f{}_{,tt} = \Delta_{,r}\,, \tag{21}$$

respectively, where $\left(u^f, v^f\right)$ is the displacement vector, B^f is the adiabatic bulk modulus, $c_f = \sqrt{B^f / \rho^f}$ is the acoustic phase velocity of the fluid in which ρ^f is the density of the fluid and

$$\Delta = \left(u^f{}_{,r} + r^{-1}\left(u^f + v^f{}_{,\theta}\right)\right), \tag{22}$$

substituting $u^f = \phi^f{}_{,r}$ and $v^f = r^{-1}\phi^f{}_{,\theta}$ and seeking the solution of Eq.(14) in the form

$$\bar{\phi}^f\left(r,\theta,t\right) = \phi^f \cos n\theta \, \exp\left\{i\left(\varsigma \bar{z} + \Omega T_a\right)\right\}, \tag{23}$$

where

$$\phi^f = A_4 J_n^1\left(\alpha_5 a x\right) \tag{24}$$

for inner fluid. In Eq.(17), $\left(\alpha_5 a\right)^2 = \Omega^2 / \bar{\rho}\,\overline{B^f}$ in which $\bar{\rho} = \rho / \rho^f$, $\overline{B^f} = B^f / \mu$, J_n^1 is the Bessel function of the first kind. If $\left(\alpha_5 a\right)^2 < 0$, the Bessel function of first kind is to be replaced by the modified Bessel function of second kind K_n . Similarly,

$$\bar{\phi}^f\left(r,\theta,t\right) = \phi^f \cos n\theta \exp\left\{i\left(\varsigma \bar{z} + \Omega T_a\right)\right\}, \tag{25}$$

where

$$\phi^f = A_5 H_n^1\left(\alpha_5 a x\right) \tag{26}$$

for outer fluid. In Eq.(17) $\left(\alpha_5 a\right)^2 = \Omega^2 / \bar{\rho}\,\bar{B}^f$. H_n^1 is the Hankel function of the first kind. If $\left(\alpha_5 a\right)^2 < 0$, then the Hankel function of first kind is to be replaced by K_n, where K_n is the modified Bessel functioning of second kind.

By substituting the expression of the displacement vector in terms of ϕ^f and the equations (21) and (23) in Eq.(20), we could express the acoustic pressure both inner and outer surface of the plate as

$$p_1{}^f = A_4 \Omega^2 \bar{\rho} J_n^1\left(\alpha_5 a x\right) \cos n\theta \exp\left\{i\left(\varsigma \bar{z} + \Omega T_a\right)\right\} \tag{27}$$

for inner fluid and

$$p_2{}^f = A_5 \Omega^2 \bar{\rho} H_n^1\left(\alpha_5 a x\right) \cos n\theta \exp\left\{i\left(\varsigma \bar{z} + \Omega T_a\right)\right\} \tag{28}$$

for outer fluid.

5. Frequency equations

Substituting the solutions given in the Eqs.(16), (19) and (27) in the boundary condition in the Eqs. (9), we obtain a system of five linear algebraic equation as follows:

$a_{57} = nJ_n\left(\alpha_4 a\right)$,

$a_{59} = -\left\{nH_n^{(1)}\left(\alpha_5 a\right) - \left(\alpha_5 a\right)H_{n+1}^{(1)}\left(\alpha_5 a\right)\right\}$,

where $E_{ij}\ \left(j = 2, 4, 6, 8, 10\right)$ can be obtained by just replacing the Bessel functions of the first kind in $E_{ij}\ \left(i = 1, 3 5, 7, 9\right)$ with those of the second kind, respectively, while $E_{ij}\ \left(i = 6, 7, 8, 9, 10\right)$ can be obtained by just replacing a in $E_{ij}\ \left(i = 1, 2, 3, 4, 5\right)$ with *b*. Now we consider the coupled free vibration problem.

6. Numerical results and discussion

The coupled free wave propagation in a homogenous isotropic generalized thermo elastic cylindrical plate immersed in water is numerically solved for the PZT-4 material. The material properties of PZT-4 are given as follows and for the purpose of numerical computation the liquid is taken as water. The material properties of PZT-4 is taken form Berlincourt et al. [21] are used for the numerical calculation is given below: $c_{11} = 13.9\times10^{10}\, N\, m^{-2}$, $c_{12} = 7.78\times10^{10}\, N\, m^{-2}$, $c_{13} = 7.43\times10^{10}\, N\, m^{-2}$, $c_{33} = 11.5\times10^{10}\, N\, m^{-2}$, $c_{44} = 2.56\times10^{10}\, N\, m^{-2}$, $c_{66} = 3.06\times10^{10}\, N\, m^{-2}$, $e_{31} = -5.2\, C\, m^{-2}$, $e_{33} = 15.1\, C\, m^{-2}$, $e_{15} = 12.7\, C\, m^{-2}$, $\varepsilon_{11} = 6.46\times10^{-9}\, C^2 N^{-1} m^{-2}$, $\varepsilon_{33} = 5.62\times10^{-9}\, C^2 N^{-1} m^{-2}$, $\rho = 7500\, Kg\, m^{-2}$ and for fluid the density $\rho^f = 1000\, Kg\, m^{-3}$, phase velocity $c_f = 1500\, m \sec^{-1}$. The velocity ratio between the fluid and solid is defined as $v_R = c_1 / c_f$.

The roots of the algebraic equation in Eq.(14) were calculated using a combination of the Birge-Vita method and Newton-Raphson method. For the present case, the simple Birge-Vita method does not work for finding the root of the algebraic equation. After obtaining the roots of the algebraic equation using the Birge-Vita method, the roots are corrected for the desired accuracy using the Newton-Raphson method. Such a combination can overcome the difficulties encountered in finding the roots of the algebraic equations of the governing equations.

The complex secular Eq.(30) contains complete information regarding wave number, phase velocity and attenuation coefficient and other propagation characteristics of the considered surface waves. In order to solve this equation we take

$$c^{-1} = v^{-1} + i\omega^{-1} q\,, \tag{31}$$

where $k = R + iq$, $R = \dfrac{\omega}{v}$ and R, q are real numbers. Here it may be noted that v and q respectively represent the phase velocity and attenuation coefficient of the waves. Upon using the representation (31) in equation (30) and various relevant relations, the complex roots $\lambda_i^2\ (i = 1, 2, 3, 4)$ of the quadratic equation (14) can be computed with the help of Secant method. The characteristics roots $\lambda_i^2\ (i = 1, 2, 3, 4)$ are further used to solve the equation (30) to obtain phase velocity (v) and attenuation coefficient (q) by using the functional iteration numerical technique as given below.

The equation (30) is of the form $F(C) = 0$ which upon using representation (31) leads to a system of two real equations $f(v,q) = 0$ and $g(v,q) = 0$. In order to apply functional iteration method, we write $v = f^*(v,q)$ and $q = g^*(v,q)$, where the functions f^* and g^* are selected in such a way that they satisfies the conditions

fluid and LMWOF denotes the longitudinal mode without fluid, 1 refer to the first mode and 2 refer the second mode in all the dispersion curves. In Figs. 1 and 2, the dispersion of dimensionless frequency Ω with the non-dimensional wave number $|\varsigma|$ is studied for the two values of the velocity ratio $v_R = 0.5$ and $v_R = 1.0$.

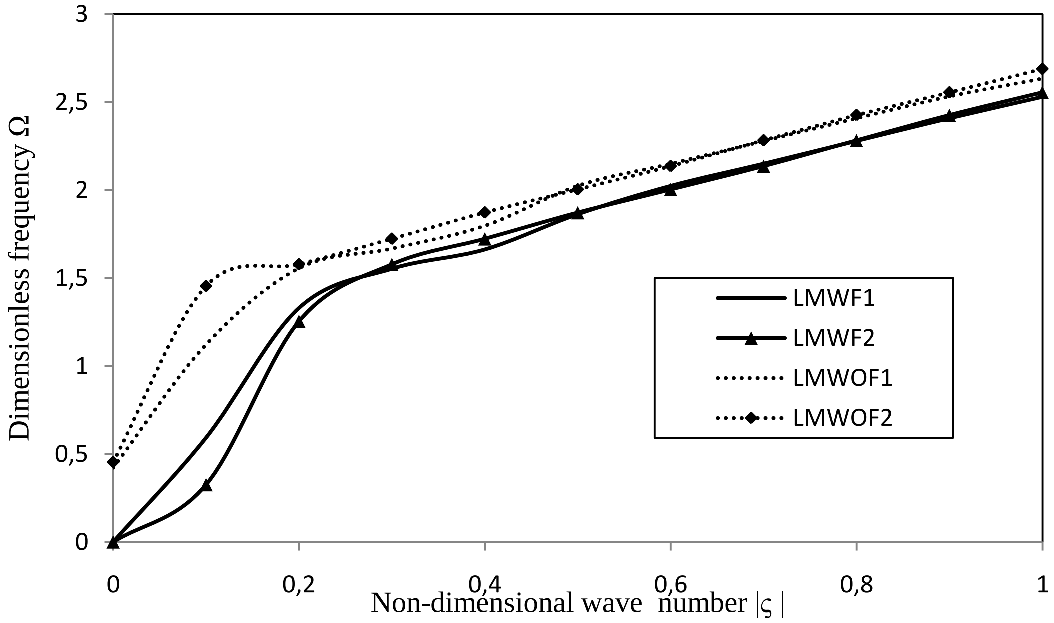

Fig. 2. Dispersion of dimensionless frequency with non-dimensional wave number of cylindrical plate for $v_R = 1.0$.

From Fig. 2, it is observed that the frequency increases linearly with increasing wave number for the low velocity ratio. There is a small deviation from the linear behavior in the current range of wave numbers in Fig. 3 for the larger velocity ratio. As the wave number increases, the longitudinal modes getting more energy than the longitudinal mode with fluid due to the dissipating nature of embedding fluid.

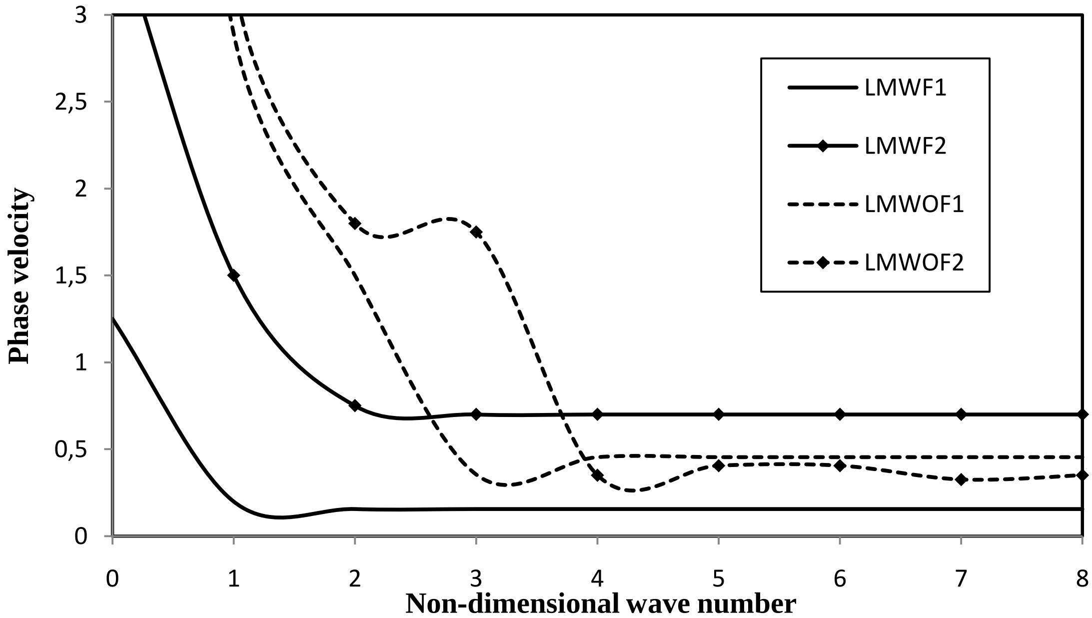

Fig. 3. Dispersion of phase velocity c with non-dimensional wave number $|\varsigma|$ of cylindrical plate for $v_R = 0.5$.

The dispersion of attenuation coefficient q with respect to the wave number $|\varsigma|$ is discussed for the two velocity ratio $v_R = 0.5$ and $v_R = 1.0$ in Figs. 5 and 6. The amplitude of displacement of the attenuation coefficient increases monotonically to attain maximum value in lower wave number and slashes down to became asymptotically linear in the remaining range of the wave number in Fig. 5. The variation of attenuation coefficient for increasing velocity ratio of longitudinal modes with and without fluid is oscillating in the maximum range of wave number as shown in Fig. 6. From Fig. 5 and Fig. 6 it is clear that the attenuation profiles exhibits high oscillating nature in the higher velocity ratio The cross over points in the vibrational modes indicates the energy transfer between the solid and fluid medium.

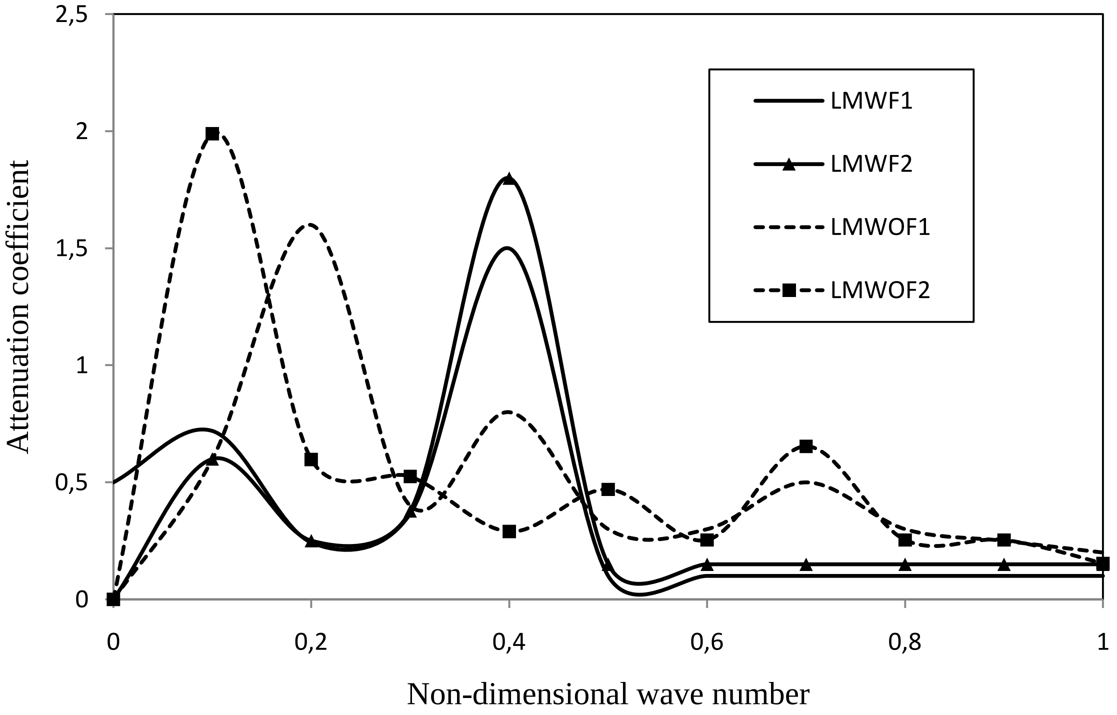

Fig. 6. Dispersion of attenuation coefficient q with non-dimensional wave number $|\varsigma|$ of cylindrical plate for $v_R = 1.0$.

7. Conclusions

The three-dimensional elasto dynamic wave propagation of a homogeneous transversely isotropic piezoelectric cylindrical plate immersed in fluid is investigated in this paper. For this problem, the governing equations of three-dimensional linear theory of elasticity have been employed with electrostatic equation and solved by the Bessel function solutions with complex arguments. The effects of the frequency, attenuation coefficient and phase velocity with respect to the non-dimensional wave number of a PZT-4 cylindrical plate is studied and are plotted as dispersion curves.

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