

GENERALIZED THERMOELASTIC WAVES IN A ROTATING RING SHAPED CIRCULAR PLATE IMMERSSED IN AN INVISCID FLUID

Rajendran Selvamani^{1*}, P. Ponnusamy²

¹Department of Mathematics Karunya University, Coimbatore, Tamil Nadu, India.

²Department of Mathematics Govt Arts College (Autonomous), Coimbatore, Tamil Nadu, India.

*e-mail: selvam1729@gmail.com

Abstract. In this paper, the generalized thermoelastic waves in a rotating ring shaped circular plate immersed in fluid are studied based on the Lord-Shulman (LS) and Green-Lindsay (GL) generalized two dimensional theory of thermoelasticity. Two displacement potential functions are introduced to uncouple the equations of motion. The frequency equations that include the interaction between the plate and fluid are obtained by the traction free boundary conditions using the Bessel function solutions. The numerical calculations are carried out for the material Zinc and the computed non-dimensional frequency, phase velocity, attenuation coefficient and relative frequency shift are plotted as the dispersion curves for the plate with thermally insulated and isothermal boundaries. The wave characteristics are found to be more dispersive and realistic in the presence of thermal relaxation time, fluid and the rotation parameter.

1. Introduction

The thermomechanical effects produced by the interaction of temperature and deformation fields are of particular importance in many modern designs including gas and steam turbines, jets and rockets, high speed airplanes and nuclear reactors. Hyperbolic heat transport has been receiving increasing attention both for theoretical and for analysis of some practical problems involving fast supply of thermal energy. The usual theory of thermal conduction, based on the Fourier's law implies an immediate response to a temperature gradient and leads to a parabolic differential equation for the evolution of the temperature. In contrast, when relaxation effects are taken in to account in the constitutive equation describing the heat flux, heat conduction equation becomes a hyperbolic equation, which implies a finite speed for heat transport. The analysis of thermally induced wave propagation of rotating circular plate immersed in an inviscid fluid medium is common place in the design of structures, atomic reactors, steam turbines, wave loading on submarine, the impact loading due to superfast train and jets and other devices operating at elevated temperature. When stress wave propagates along embedded structures, they are constrained between its geometric boundaries and they undergo multiple reflections. A complex mixture of constructive and destructive interferences arises from successive reflections, refractions and mode conversion due to the interaction between waves and embedding fluid medium. So, this type of study may be used in applications involving nondestructive testing (NDT), qualitative nondestructive evaluation (QNDE) of large diameter pipes and health monitoring of other ailing infrastructures in addition to check and verify the validity of FEM and BEM for such problems.

The generalized theory of thermoelasticity was developed by Lord-Shulman [1] involving one relaxation time for isotropic homogeneous media, which is called the first generalization to the coupled theory of elasticity. These equations determine the finite speeds

of propagation of heat and displacement distributions, the corresponding equations for an isotropic case were obtained by Dhaliwal and Sherief [2]. The second generalization to the coupled theory of elasticity is what is known as the theory of thermoelasticity, with two relaxation times or the theory of temperature-dependent thermoelectricity. A generalization of this inequality was proposed by Green and Laws [3]. Green and Lindsay [4] obtained an explicit version of the constitutive equations. These equations were also obtained independently by Suhubi [5]. This theory contains two constants that act as relaxation times and modify not only the heat equations, but also all the equations of the coupled theory. The classical Fourier's law of heat conduction is not violated if the medium under consideration has a center of symmetry. Erbay and Suhubi [6] studied the longitudinal wave propagation in a generalized thermoplastic infinite cylinder and obtained the dispersion relation for a constant surface temperature of the cylinder. Ponnusamy and Selvamani [7] have studied the dispersion analysis of generalized magneto-thermoelastic waves in a transversely isotropic cylindrical panel using the wave propagation approach. Later, Selvamani [8] obtained mathematical modeling and analysis for damping of generalized thermoelastic waves in a homogeneous isotropic plate. Sharma and Pathania [9] investigated the generalized wave propagation in circumferential curved plates. Modeling of circumferential wave in cylindrical thermoelastic plate with voids was discussed by Sharma and Kaur [10]. Ashida and Tauchert [11] presented the temperature and stress analysis of an elastic circular cylinder in contact with heated rigid stamps. Later, Ashida [12] analyzed thermally - induced wave propagation in a piezoelectric plate. Tso and Hansen [13] have studied the wave propagation through cylinder/plate junctions. Heyliger and Ramirez [14] analyzed the free vibration characteristics of laminated circular piezoelectric plates and disc by using a discrete-layer model of the weak form of the equations of periodic motion. Thermal deflection of an inverse thermoelastic problem in a thin isotropic circular plate was presented by Gaikward and Deshmukh [15]. The theory of elastic material with rotation is plays a vital role in civil, architecture, aeronautical and marine engineering. Body wave propagation in rotating thermoelastic media was investigated by Sharma and Grover [16]. The effect of rotation, magneto field, thermal relaxation time and pressure on the wave propagation in a generalized visco elastic medium under the influence of time harmonic source is discussed by Abd-Alla and Bayones [17]. The propagation of waves in conducting piezoelectric solid is studied for the case when the entire medium rotates with a uniform angular velocity by Wauer [18]. Roychoudhuri and Mukhopadhyay studied the effect of rotation and relaxation times on plane waves in generalized thermo-visco-elasticity [19]. Gamer [20] discussed the elastic-plastic deformation of the rotating solid disk. Lam [21] studied the frequency characteristics of a thin rotating cylindrical shell using general differential quadrature method. Rama Rao [22] was studied the acoustic of fluid filled boreholes with pipes. Here he developed a three dimensional elasto dynamic equation for axis symmetric waves of pipes immersed inside fluid filled boreholes in infinite elastic spaces. Nagy [23] was investigated longitudinal guided wave propagation in a transversely isotropic rod immersed in fluid based on the superposition of partial waves. Ahamed [24] discussed the guided waves in a transversely isotropic cylinder immersed in fluid. Later, Ponnusamy [25] has studied the wave propagation of generalized thermo elastic solid cylinder of arbitrary cross section immersed in fluid using Fourier collocation method Ahamed [26] discussed the guided waves in a transversely isotropic plate immersed in fluid. Chan [27] studied the Lamb waves in highly attenuative plastic plate.

In this problem, the in-plane vibration of generalized thermoelastic rotating thin ring shaped circular plate immersed in an inviscid fluid medium composed of homogeneous isotropic material is studied. The solutions to the equations of motion for an isotropic medium is obtained by using the two dimensional theory of elasticity and Bessel function solutions. The numerical calculations are carried out for the material Zinc. The computed non-

dimensional frequency, phase velocity, attenuation coefficient and relative frequency shift are plotted in dispersion curves for thermally insulated and isothermal boundary of the plate.

2. Formulation of the problem

We consider a thin homogeneous, isotropic, thermally conducting elastic plate of radius R with uniform thickness h and temperature T_0 in the undisturbed state. The system displacements and stresses are defined in polar coordinates (r, θ, z) .

The two dimensional stress equations of motion in the absence of body force for a linearly elastic rotating medium are

$$\begin{aligned}\sigma_{rr,r} + r^{-1}\sigma_{r\theta,\theta} + \sigma_{rz,z} + r^{-1}(\sigma_{rr} - \sigma_{\theta\theta}) + \rho(\vec{\Omega} \times (\vec{\Omega} \times \vec{u}) + 2\vec{\Omega} \times \vec{u}_{,t}) &= \rho u_{,tt}, \\ \sigma_{r\theta,r} + r^{-1}\sigma_{\theta\theta,\theta} + \sigma_{rz,z} + \sigma_{\theta z,z} + 2r^{-1}\sigma_{r\theta} &= \rho v_{,tt}, \\ \sigma_{rz,r} + r^{-1}\sigma_{\theta z,\theta} + r^{-1}\sigma_{r\theta} + \rho(\vec{\Omega} \times (\vec{\Omega} \times \vec{u}) + 2\vec{\Omega} \times \vec{u}_{,t}) &= \rho w_{,tt}\end{aligned}\quad (1a)$$

The heat conduction equation is

$$k(T_{,rr} + r^{-1}T_{,r} + r^{-2}T_{,\theta\theta}) - \rho c_v(T_{,t} + \tau_0 T_{,tt}) = \beta T_0 \left(\frac{\partial}{\partial t} + \tau_0 \delta_{1k} \frac{\partial^2}{\partial t^2} \right) [e_{rr} + e_{\theta\theta}], \quad (1b)$$

where ρ is the mass density, c_v is the specific heat capacity, k is the thermal conductivity, T_0 is the reference temperature, the displacement equation of motion has the additional terms with a time dependent centripetal acceleration $\vec{\Omega} \times (\vec{\Omega} \times \vec{u})$ and $2\vec{\Omega} \times \vec{u}_{,t}$, where $\vec{u} = (u, 0, w)$ is the displacement vector and $\vec{\Omega} = (0, \Omega, 0)$ is the angular velocity, the comma notation used in the subscript denotes the partial differentiation with respect to the variables

The stress strain relation is given by

$$\begin{aligned}\sigma_{rr} &= \lambda(e_{rr} + e_{\theta\theta}) + 2\mu e_{rr} - \beta(T + \delta_{2k}\tau_1 T_{,t}), \\ \sigma_{\theta\theta} &= \lambda(e_{rr} + e_{\theta\theta}) + 2\mu e_{\theta\theta} - \beta(T + \delta_{2k}\tau_1 T_{,t}), \\ \sigma_{r\theta} &= 2\mu e_{r\theta}, \\ \sigma_{rz} &= 2\mu e_{rz},\end{aligned}\quad (2)$$

where e_{ij} are the strain components, $\beta = (3\lambda + 2\mu)\alpha_T$ is the thermal stress coefficient, α_T is the coefficient of linear thermal expansion, T is the temperature, t is the time, λ and μ are Lamé constants. τ_0 and τ_1 are the thermal relaxation times and the comma notation is used for spatial derivatives. Here δ_{ij} is the Kronecker delta function. In addition, we can replace $k=1$ for LS theory and $k=2$ for GL theory. The thermal relaxation times τ_0 and τ_1 satisfies the inequalities $\tau_0 \geq \tau_1 \geq 0$ for GL theory only.

The strain e_{ij} are related to the displacements are given by

$$e_{rr} = u_{,r}, e_{\theta\theta} = r^{-1}(u + v_{,\theta}), \quad 2e_{r\theta} = v_{,r} - r^{-1}(v - u_{,\theta}), \quad 2e_{rz} = (u_{,z} + w_{,r}), \quad (3)$$

in which u and v are the displacement components along radial and circumferential

directions respectively. $\sigma_{rr}, \sigma_{\theta\theta}$ are the normal stress components and $\sigma_{r\theta}, \sigma_{\theta z}, \sigma_{zr}$ are the shear stress components, $e_{rr}, e_{\theta\theta}$ are normal strain components and $e_{r\theta}, e_{\theta z}, e_{zr}$ are shear strain components. The comma in the subscripts denotes the partial differentiation with respect to the variables.

Substituting Eqs. (3) and (2) in Eqs. (1), the following displacement equations of motions are obtained as follows:

$$\begin{aligned}
 & (\lambda + 2\mu)(u_{,rr} + r^{-1}u_{,r} - r^{-2}u) + \mu r^{-2}u_{,\theta\theta} + r^{-1}(\lambda + \mu)v_{,r\theta} + r^{-2}(\lambda + 3\mu)v_{,\theta} \\
 & + \rho(\Omega^2 u + 2\Omega w_{,t}) - \beta(T_{,r} + \delta_{2k}\tau_1 T_{,rt}) = \rho u_{,tt}, \\
 & \mu(v_{,rr} + r^{-1}v_{,r} - r^{-2}v) + r^{-2}(\lambda + 2\mu)v_{,\theta\theta} + r^{-2}(\lambda + 3\mu)u_{,\theta} + r^{-1}(\lambda + \mu)u_{,r\theta} \\
 & - \beta(T_{,\theta} + \delta_{2k}\tau_1 T_{,\theta t}) = \rho v_{,tt}, \\
 & (\lambda + \mu)u_{,rz} + \mu r^{-1}(\lambda + \mu)v_{,\theta z} + \mu(w_{,rr} + r^{-1}w_{,r} + r^{-2}w_{,\theta\theta}) \\
 & - \beta(T_{,z} + \delta_{2k}\tau_1 T_{,zt}) + \rho(\Omega^2 w + 2\Omega u_{,t}) = \rho w_{,tt}, \\
 & k(T_{,rr} + r^{-1}T_{,r} + r^{-2}T_{,\theta\theta}) - \rho c_v(T_{,t} + \tau_0 T_{,tt}) = \beta T_0 \left(\frac{\partial}{\partial t} + \tau_0 \delta_{1k} \frac{\partial^2}{\partial t^2} \right) [u_{,r} + r^{-1}(u + v_{,\theta})]. \quad (4)
 \end{aligned}$$

In an inviscid fluid-solid interface, the perfect-slip boundary condition allows discontinuity in planar displacement components. That is, the radial component of displacement of the fluid and solid must be equal and the circumferential, longitudinal components are discontinuous at the interface. The above coupled partial differential equations is also subjected to the following non-dimensional boundary conditions at the surfaces $r = a, b$.

(i). Stress free inner boundary conditions

$$(\sigma_{rr} + p_1^f) = (\sigma_{r\theta}) = (\sigma_{rz}) = (u - u^f) = 0. \quad (5a)$$

(ii). Stress free outer boundary conditions

$$(\sigma_{rr} + p_2^f) = (\sigma_{r\theta}) = (\sigma_{rz}) = (u - u^f) = 0. \quad (5b)$$

(iii). Thermal boundary conditions

$$T_{,r} + hT = 0, \quad (5c)$$

where h is the surface heat transfer coefficient. Here $h \rightarrow 0$ corresponds to a thermally insulated surface and $h \rightarrow \infty$ refers to an isothermal one.

2.1. Lord-Shulman (LS) theory. The three dimensional rate dependent temperatures with one relaxation time called the Lord-Shulman theory of thermoelasticity is obtained by replacing $k = 1$ in the heat conduction equation of Eq. (1b) in the absence of rotation.

The heat conduction equation is simplified as

$$k \left(T_{,rr} + \frac{1}{r} T_{,r} + \frac{1}{r^2} T_{,\theta\theta} \right) = \rho C_v [T_{,t} + \tau_0 T_{,tt}] + \beta T_0 \left[\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right] (e_{rr} + e_{\theta\theta}), \quad (6a)$$

The stress –strain relation is replaced by

$$\begin{aligned}\sigma_{rr} &= \lambda(e_{rr} + e_{\theta\theta}) + 2\mu e_{\theta\theta} - \beta(T), \\ \sigma_{\theta\theta} &= \lambda(e_{rr} + e_{\theta\theta}) + 2\mu e_{\theta\theta} - \beta(T), \\ \sigma_{r\theta} &= 2\mu e_{r\theta}.\end{aligned}\tag{6b}$$

Upon using these relations in Eq.1 we can get the following displacement equation

$$\begin{aligned}(\lambda + 2\mu)(u_{,rr} + r^{-1}u_{,r} - r^{-2}u) + r^{-2}\mu u_{,\theta\theta} + r^{-1}(\lambda + \mu)v_{,r\theta} + r^{-2}(\lambda + 3\mu)v_{,\theta} - \beta(T) &= \rho u_{,tt}, \\ (\mu)(v_{,rr} + r^{-1}v_{,r} - r^{-2}v) + r^{-1}(\lambda + \mu)u_{,r\theta} + r^{-2}(\lambda + 3\mu)u_{,\theta} + r^{-2}(\lambda + 3\mu)v_{,\theta} \\ &+ r^{-2}(\lambda + 2\mu)v_{,\theta\theta} - \beta(T) = \rho v_{,tt}, \\ (\lambda + \mu)u_{,rz} + r^{-1}(\lambda + \mu)v_{,\theta z} + \mu(w_{,rr} + r^{-1}w_{,r} + r^{-2}w_{,\theta\theta}) \\ &- \beta(T) + \rho(\Omega^2 w + 2\Omega u_{,t}) = \rho w_{,tt},\end{aligned}\tag{6c}$$

The symbols and notations having the same meaning are defined in earlier sections. Since the heat conduction equation in this theory is of hyperbolic wave type, it is automatically ensures that the finite speeds of propagation for heat and elastic waves.

2.2. Green-Lindsay (GL) theory. The second generalization to the coupled thermoelasticity with two relaxation times called Green-Lindsay theory of thermoelasticity is obtained by setting $k=2$ in the heat conduction Eq.(1b)

The heat conduction equation is simplified as

$$k\left(T_{,rr} + \frac{1}{r}T_{,r} + \frac{1}{r^2}T_{,\theta\theta}\right) = \rho C_v [T_{,t} + \tau_0 T_{,tt}] + \beta T_0 \frac{\partial}{\partial t}(e_{rr} + e_{\theta\theta}).\tag{7a}$$

The stress –strain relation is replaced by

$$\begin{aligned}\sigma_{rr} &= \lambda(e_{rr} + e_{\theta\theta}) + 2\mu e_{\theta\theta} - \beta(T + \tau_1 T_{,t}), \\ \sigma_{\theta\theta} &= \lambda(e_{rr} + e_{\theta\theta}) + 2\mu e_{\theta\theta} - \beta(T + \tau_1 T_{,t}), \\ \sigma_{r\theta} &= 2\mu e_{r\theta}.\end{aligned}\tag{7b}$$

Substituting these relations in Eq.1, the displacement equation is reduced as

$$\begin{aligned}(\lambda + 2\mu)(u_{,rr} + r^{-1}u_{,r} - r^{-2}u) + r^{-2}\mu u_{,\theta\theta} + r^{-1}(\lambda + \mu)v_{,r\theta} + r^{-2}(\lambda + 3\mu)v_{,\theta} \\ - \beta(T_{,r} + \tau_1 T_{,rt}) = \rho u_{,tt}, \\ (\mu)(v_{,rr} + r^{-1}v_{,r} - r^{-2}v) + r^{-1}(\lambda + \mu)u_{,r\theta} + r^{-2}(\lambda + 3\mu)u_{,\theta} + r^{-2}(\lambda + 3\mu)v_{,\theta} \\ + r^{-2}(\lambda + 2\mu)v_{,\theta\theta} - \beta(T_{,\theta} + \tau_1 T_{,\theta t}) = \rho v_{,tt},\end{aligned}$$

$$\begin{aligned}
& (\lambda + \mu)u_{,rz}\mu + r^{-1}(\lambda + \mu)v_{,\theta z} + \mu(w_{,rr} + r^{-1}w_{,r} + r^{-2}w_{,\theta\theta}) \\
& -\beta(T_{,z} + \tau_1 T_{,zt}) + \rho(\Omega^2 w + 2\Omega u_{,t}) = \rho w_{,tt}, \quad (7c)
\end{aligned}$$

where the symbols and notations are defined in the previous sections. In view of available experimental evidence in favor of the finiteness of heat propagation speeds, the generalized thermoelasticity theories are supposed to be more realistic than the conventional theory in dealing with practical problems involving very large heat fluxes and/or short time intervals like those occurring in laser units and energy channels.

3. Solutions of solid medium

The Eqs. (4) are the coupled partial differential equations with two displacements and heat conduction components. To uncouple the Eqs. (4), we can assume the vibration and displacements along the axial direction z equal to zero. Hence assuming the solutions of the Eqs. (4) in the form

$$u(r, \theta, t) = \sum_{n=0}^{\infty} \varepsilon_n \left[(\phi_{n,r} + r^{-1}\psi_{n,\theta}) + (\bar{\phi}_{n,r} + r^{-1}\bar{\psi}_{n,\theta}) \right] \exp\{i(p\theta - \omega t)\}, \quad (8a)$$

$$v(r, \theta, t) = \sum_{n=0}^{\infty} \varepsilon_n \left[(r^{-1}\phi_{n,\theta} - \psi_{n,r}) + (r^{-1}\bar{\phi}_{n,\theta} - \bar{\psi}_{n,r}) \right] \exp\{i(p\theta - \omega t)\}, \quad (8b)$$

$$w(r, \theta, t) = (i/a) \sum_{n=0}^{\infty} \varepsilon_n [W_n + \bar{W}_n] \exp\{i(p\theta - \omega t)\}, \quad (8c)$$

$$T(r, \theta, t) = (\lambda + 2\mu/\beta a^2) \sum_{n=0}^{\infty} \varepsilon_n (T_n + \bar{T}_n) \exp\{i(p\theta - \omega t)\}, \quad (8d)$$

where $\varepsilon_n = \frac{1}{2}$ for $n=0$, $\varepsilon_n = 1$ for $n \geq 1$, $i = \sqrt{-1}$, ω is the frequency, $\phi_n(r, \theta)$, $\psi_n(r, \theta)$, $T_n(r, \theta)$, $\bar{\phi}_n(r, \theta)$, $\bar{\psi}_n(r, \theta)$, $\bar{T}_n(r, \theta)$, and $\bar{W}_n(r, \theta)$ are the displacement potentials. Introducing the dimensionless quantities such as $T_a = t\sqrt{\mu/\rho}/a$, $x = r/a$, $c_1^2 = (\lambda + 2\mu)/\rho$, $\xi^2 = \rho\omega^2 R^2/\mu$, $\alpha' = c_1 a/K$, $\varpi^2 = \omega^2 R^2/c_1^2$, $\Gamma = \frac{\rho\Omega^2 R^2}{2 + \bar{\lambda}}$, $\varepsilon_1 = \frac{T_0 a}{\rho^2 c_v c_1 K} \beta^2$, $\varepsilon_2 = \frac{c_1^2}{c_v K} \tau$, and

$\varepsilon_3 = \frac{c_1}{a} \eta$ and using Eqs. (8) in Eq. (4), we obtain

$$(A\nabla^6 + B\nabla^4 + C\nabla^2 + D)(\phi_n, w_n, T_n) = 0, \quad (9)$$

where

$$A = 1, \quad B = \varpi^2(1 + \varepsilon_1 \varepsilon_3) + \varepsilon_2 - i\varpi(\alpha' + \varepsilon_1), \quad C = \varpi^2(\varepsilon_2 - i\alpha'\varpi), \quad (10)$$

and

$$\begin{aligned}
D = & i\Omega\alpha'(1 + i\chi_3\Omega) \left(-\varsigma^2\chi_4(1 + \bar{\lambda}) + (\Omega^2 - \chi_4\varsigma^2)\varsigma^2 + (\Omega^2 - \varsigma^2) \right) \\
& + (\chi_2 - i\Omega\alpha')(\Omega^2 - \chi_4\varsigma^2)(\Omega^2 - \varsigma^2),
\end{aligned}$$

$$\left(\nabla^2 + (2 + \bar{\lambda})\varpi^2 - \zeta^2 + \Gamma\right) \psi_n = 0, \quad (11)$$

where $\nabla^2 \equiv \partial^2/\partial x^2 + x^{-1}\partial/\partial x + x^{-2}\partial^2/\partial\theta^2$.

The parameters defined in Eq. (10) namely, ε_1 couples the equations corresponding to the elastic wave propagation and the heat conduction which is called the coupling factor; the coefficient ε_2 , which is introduced by the theory of generalized thermoelasticity, may render the governing system of equations hyperbolic. The parameter ε_3 is the coefficient of the term indicating the difference between empirical and thermodynamic temperatures.

Solving the partial differential Eq. (6), the solutions for symmetric mode is obtained as

$$\phi_n = \sum_{i=1}^2 \left[A_{in} J_n(\alpha_i ax) + B_{in} Y_n(\alpha_i ax) \right] \cos n\theta, \quad (12a)$$

$$T_n = \sum_{i=1}^2 d_i \left[A_{in} J_n(\alpha_i ax) + B_{in} Y_n(\alpha_i ax) \right] \cos n\theta, \quad (12b)$$

and the solution for the anti symmetric mode $\bar{\phi}_n$ is obtained by replacing $\cos n\theta$ by $\sin n\theta$ in Eqs. (12), we get

$$\bar{\phi}_n = \sum_{i=1}^2 \left[\bar{A}_{in} J_n(\alpha_i ax) + \bar{B}_{in} Y_n(\alpha_i ax) \right] \sin n\theta, \quad (13a)$$

$$\bar{T}_n = \sum_{i=1}^2 d_i \left[\bar{A}_{in} J_n(\alpha_i ax) + \bar{B}_{in} Y_n(\alpha_i ax) \right] \sin n\theta, \quad (13b)$$

Equation (11) is a Bessel equation with its possible solution as

$$\bar{\psi} = \begin{cases} A_3 J_n(\alpha_3 ax) + B_3 Y_n(\alpha_3 ax) & \alpha_3 ax > 0 \\ A_3 a^n + B_3 a^{-n} & \alpha_3 ax = 0, \\ A_3 I_n(\alpha_3 ax) + B_3 K_n(\alpha_3 ax) & \alpha_3 ax < 0 \end{cases} \quad (14)$$

where J_n and Y_n are Bessel functions of the first and second kinds respectively while, I_n and K_n are modified Bessel functions of first and second kinds respectively. (A_i, B_i) $i = 1, 2, 3$ are the arbitrary constants. Generally $\alpha_3 ax \neq 0$, so that the situation $\alpha_3 ax \neq 0$ will not be discussed in the following. For convenience, we will pay attention only to the case of $\alpha_3 ax > 0$ in what follows, and the derivation for the case of $\alpha_3 ax < 0$ is similar.

Solving Eq. (11), we obtain

$$\psi_n = \left[A_{3n} J_n(\alpha_3 ax) + B_{3n} Y_n(\alpha_3 ax) \right] \sin n\theta \quad (15a)$$

for symmetric mode, and for the antisymmetric mode $\bar{\psi}_n$ is obtained from Eq. (15a) by replacing $\sin n\theta$ by $\cos n\theta$.

$$\bar{\psi}_n = \left[\bar{A}_{3n} J_n(\alpha_3 ax) + \bar{B}_{3n} Y_n(\alpha_3 ax) \right] \cos n\theta, \quad (15b)$$

where $(\alpha_3 a)^2 = (2 + \bar{\lambda})\varpi^2 - \zeta^2 + \Gamma$. If $(\alpha_i a)^2 < 0$ ($i = 1, 2, 3$), then the Bessel functions J_n and

Y_n are replaced by the modified Bessel function I_n and K_n respectively.

$$\bar{\psi} = [A_3 J_n(\alpha_3 a x) + B_3(\alpha_3 a x)], \quad (16)$$

where $(\alpha_3 a)^2 = \varpi^2$. The constant defined in the Eqs. (12) is calculated from the equation $d_i = (\varpi^2 - (2 + \bar{\lambda})(\alpha_i a)^2)$.

4. Solution of fluid medium

In cylindrical coordinates, the acoustic pressure and radial displacement equation of motion for an in viscid fluid are of the form

$$p^f = -B^f (u^f_{,r} + r^{-1}(u^f + v^f_{,\theta})) \quad (17)$$

and

$$c_f^{-2} u^f_{,tt} = \Delta_{,r}, \quad (18)$$

respectively, where (u^f, v^f) is the displacement vector, B^f is the adiabatic bulk modulus, $c_f = \sqrt{B^f / \rho^f}$ is the acoustic phase velocity of the fluid in which ρ^f is the density of the fluid and

$$\Delta = (u^f_{,r} + r^{-1}(u^f + v^f_{,\theta})), \quad (19)$$

substituting $u^f = \phi^f_{,r}$ and $v^f = r^{-1} \phi^f_{,\theta}$ and seeking the solution of Eq. (18) in the form

$$\bar{\phi}^f(r, \theta, t) = \phi^f \exp\{i(p\theta - \omega t)\}, \quad (20)$$

where

$$\phi^f = A_4 J_n(\delta a x) \quad (21)$$

for inner fluid. In Eq. (21), $(\delta a)^2 = \varpi^2 / \bar{\rho}^f \bar{B}^f$ in which $\bar{\rho} = \rho / \rho^f$, $\bar{B}^f = B^f / \mu$, J_n is the Bessel function of the first kind. If $(\delta a)^2 < 0$, the Bessel function of first kind is to be replaced by the modified Bessel function of second kind K_n . Similarly,

$$\bar{\phi}^f(r, \theta, t) = \phi^f \exp\{i(p\theta - \omega t)\}, \quad (22)$$

where

$$\phi^f = A_5 H_n(\delta a x) \quad (23)$$

for outer fluid. In Eq. (23) $(\delta a)^2 = \varpi^2 / \bar{\rho} \bar{B}$. H_n is the Hankel function of the first kind. If $(\delta a)^2 < 0$, then the Hankel function of first kind is to be replaced by K_n , where K_n is the modified Bessel functioning of second kind.

By substituting the expression of the displacement vector in terms of ϕ^f and the

Eqs. (20) and (22) in Eq. (17), we could express the acoustic pressure both inner and outer surface of the ring as

$$p_1^f = A_4 \varpi^2 \bar{\rho} J_n(\delta ax) \exp\{i(p\theta - \omega t)\} \quad (24)$$

for inner fluid and

$$p_2^f = A_5 \varpi^2 \bar{\rho} H_n(\delta ax) \exp\{i(p\theta - \omega t)\} \quad (25)$$

for outer fluid.

4.1. Flexural mode. Expanded expressions for the stress and displacement at any point can be derived in terms of potential functions. Upon using the potential functions defined in Eqs. (8) in to Eqs. (5) results in the following equations

$$|E_{ij}| = 0 \quad i, j = 1, 2, \dots, 8. \quad (26)$$

Equation (26) represents the frequency equation for flexural wave propagating along an isotropic circular plate immersed in fluid.

4.2. Longitudinal mode. Longitudinal waves are axially symmetric waves characterized by the presence of displacement component in the radial and axial directions. That is, for the longitudinal wave travelling along the axis, the displacement field is independent of θ coordinate and is of the form $(u, 0, 0)$. This mode of wave propagation corresponds to $n = 0$ in Eqs. (12)-(13) resulting

$$\det(E_{ij}) = \begin{vmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{vmatrix} = 0. \quad (27)$$

Equation (27) is the frequency equation for longitudinal waves propagating along an isotropic circular plate immersed in fluid.

5. Frequency equations

In this section we shall derive the secular equation for the three dimensional vibrations circular plate subjected to stress free boundary conditions at the upper and lower surfaces at $r = a, b$:

$$|E_{ij}| = 0 \quad i, j = 1, 2, \dots, 8 \quad (28)$$

$$E_{11} = (2 + \bar{\lambda})(ip) \left((nJ_n(\alpha_1 ax) + (\alpha_1 ax)J_{n+1}(\alpha_1 ax)) - (ip)((\alpha_1 ax)^2 R^2 - n^2)J_n(\alpha_1 ax) \right) \\ + \bar{\lambda} \left((ip)n(n-1)(J_n(\alpha_1 ax) - (\alpha_1 ax)J_{\delta+1}(\alpha_1 ax)) \right) - \beta T(i\omega)\eta_2 d_1(ax)^2,$$

$$E_{13} = (2 + \bar{\lambda})(ip) \left((nJ_n(\alpha_2 ax) + (\alpha_2 ax)J_{n+1}(\alpha_2 ax)) - (ip)((\alpha_2 ax)^2 R^2 - n^2)J_n(\alpha_2 ax) \right) \\ + \bar{\lambda}(ip) \left(n(n-1)(J_n(\alpha_2 ax) - (\alpha_2 ax)J_{\delta+1}(\alpha_2 ax)) \right) - \beta T(i\omega)\eta_2 d_2(ax)^2,$$

$$E_{15} = (2 + \bar{\lambda}) \left((n(n-1)J_n(\alpha_3 ax) - (\alpha_3 ax)J_{n+1}(\alpha_3 ax)) + \bar{\lambda}(n(n-1)J_n(\alpha_3 ax) - (\alpha_3 ax)J_{n+1}(\alpha_3 ax)) \right),$$

$$E_{16} = \varpi^2 \bar{\rho}(ax)^2 (nJ_n(\delta ax) - (\delta ax)J_{n+1}(\delta ax)),$$

$$E_{21} = nJ_n(\alpha_1 ax) - (\alpha_1 ax)J_{n+1}(\alpha_1 ax), \quad E_{23} = nJ_n(\alpha_2 ax) - (\alpha_2 ax)J_{n+1}(\alpha_2 ax),$$

$$E_{25} = nJ_n(\alpha_3 ax), \quad E_{27} = nJ_n(\delta ax) - (\delta ax)J_{n+1}(\delta ax),$$

$$E_{31} = 2n(n-1)J_n(\alpha_1 ax) - 2n(\alpha_1 ax)J_{n+1}(\alpha_1 ax),$$

$$E_{33} = 2n(n-1)J_n(\alpha_2 ax) - 2n(\alpha_2 ax)J_{n+1}(\alpha_2 ax),$$

$$E_{35} = (ip)(2n(n-1)J_n(\alpha_3 ax) - 2(\alpha_3 ax)J_{n+1}(\alpha_3 ax)) + (ip)((\alpha_3 ax)^2 - n^2)J_n(\alpha_3 ax),$$

$$E_{37} = 0, \quad E_{41} = d_1(nJ_n(\alpha_1 ax) - (\alpha_1 ax)J_{n+1}(\alpha_1 ax) + hJ_n(\alpha_1 ax)),$$

$$E_{43} = d_2(nJ_n(\alpha_2 ax) - (\alpha_2 ax)J_{n+1}(\alpha_2 ax) + hJ_n(\alpha_2 ax)),$$

$$E_{45} = 0, \quad E_{47} = 0, \quad E_{58} = \varpi^2 \bar{\rho}(bx)^2(nH_n(\delta ax) - (\delta ax)H_{n+1}(\delta ax)),$$

Obviously E_{ij} ($j=2,4,6$) can be obtained by just replacing the Bessel functions of the first kind in E_{ij} ($i=1,3,5$) with those of the second kind, respectively, while E_{ij} ($i=5,6,7,8$) can be obtained by just replacing a in E_{ij} ($i=1,2,3,4$) with b .

6. Numerical results and discussion

The coupled generalized thermoelastic waves in a homogenous isotropic rotating circular plate immersed in an inviscid fluid is numerically solved for the Zinc material. The material properties of Zinc are given as follows and for the purpose of numerical computation the fluid is taken as water.

For the solid:

$$\rho = 7.14 \times 10^3 \text{ kg m}^{-3}, \quad T_0 = 296 \text{ K}, \quad K = 1.24 \times 10^2 \text{ W m}^{-1} \text{ deg}^{-1}, \quad \mu = 0.508 \times 10^{11} \text{ Nm}^{-2}, \\ \beta = 5.75 \times 10^6 \text{ Nm}^{-2} \text{ deg}^{-1}, \quad \epsilon_1 = 0.0221, \quad \lambda = 0.385 \times 10^{11} \text{ Nm}^{-2} \text{ and } C_v = 3.9 \times 10^2 \text{ J kg}^{-1} \text{ deg}^{-1}.$$

For the fluid:

$$p^f = 1000 \text{ kg m}^{-3}, \quad c_f = 1500 \text{ ms}^{-1}, \text{ the rotational speed } \Omega = 1.5 \text{ and the values of the thermal relaxation times are calculated from Chandrasekharaiah [28] as } \tau_0 = 0.05 \text{ and } \tau_1 = 0.075.$$

The roots of the algebraic equation in Eq. (9) were calculated using a combination of the Birge-Vita method and Newton-Raphson method. For the present case, the simple Birge-Vita method does not work for finding the root of the algebraic equation. After obtaining the roots of the algebraic equation using the Birge-Vita method, the roots are corrected for the desired accuracy using the Newton-Raphson method. Such a combination can overcome the difficulties encountered in finding the roots of the algebraic equations of the governing equations.

The complex secular Eq. (28) contains complete information regarding wave number, phase velocity and attenuation coefficient and other propagation characteristics of the considered surface waves. In order to solve this equation we take

$$c^{-1} = v^{-1} + i\omega^{-1}q, \quad (29)$$

where $k = R + iq$, $R = \frac{\omega}{v}$ and R, q are real numbers. Here it may be noted that v and q respectively represent the phase velocity and attenuation coefficient of the waves. Upon using the representation (29) in Eq. (28) and various relevant relations, the complex roots $\alpha_i^2 (i=1,2,3)$ of the quadratic Eq. (9) can be computed with the help of Secant method. The characteristics roots $\alpha_i^2 (i=1,2,3)$ are further used to solve the Eq. (28) to obtain phase velocity (v) and attenuation coefficient (q) by using the functional iteration numerical technique as given below.

The Eq. (28) is of the form $F(C) = 0$ which upon using representation (29) leads to a system of two real equations $f(v, q) = 0$ and $g(v, q) = 0$. In order to apply functional iteration method, we write $v = f^*(v, q)$ and $q = g^*(v, q)$, where the functions f^* and g^* are selected in such a way that they satisfies the conditions

$$\left| \frac{\partial f^*}{\partial v} \right| + \left| \frac{\partial g^*}{\partial q} \right| < 1, \left| \frac{\partial g^*}{\partial v} \right| + \left| \frac{\partial f^*}{\partial q} \right| < 1 \quad (30)$$

for all v, q in the neighborhood of the roots. If (v_0, q_0) is the initial approximation of the root, then we construct a successive approximation according to the formulae

$$\begin{aligned} v_1 &= f^*(v_0, q_0) & q_1 &= g^*(v_1, q_0) \\ v_2 &= f^*(v_1, q_1) & q_2 &= g^*(v_2, q_1) \\ v_3 &= f^*(v_2, q_2) & q_3 &= g^*(v_3, q_2) \\ &\dots\dots\dots & \dots\dots\dots \\ v_n &= f^*(v_n, q_n) & q_n &= g^*(v_{n+1}, q_n) \end{aligned} \quad (31)$$

The sequence $\{v_n, q_n\}$ of approximation to the root will converge to the actual value (v_0, q_0) of the root provided, (v_0, q_0) lies in the neighborhood of the actual root. For the initial value $c = c_0 = (v_0, q_0)$, the roots $\alpha_i (i=1,2,3)$ are computed from the Eq. (9) by using Secant method for each value of the wave number k for assigned frequency. The values of $\alpha_i (i=1,2,3)$ so obtained are then used in the Eq. (28) to obtained the current values of v and q each time which are further used to generate the Eq. (31). This process is terminated as and when the condition $|v_{n+1} - v_n| < \varepsilon$, ε being arbitrary small number to be selected at random to achieve the accuracy level, is satisfied. The procedure is continually repeated for different values of the wave number (k) to obtain the corresponding values of the phase velocity (c) and attenuation coefficient (q).

The dispersion curves of non-dimensional frequency, phase velocity, attenuation and relative frequency shift of a generalized thermoelastic circular plate immersed fluid and in space with wave number are plotted for thermally insulated and isothermal boundaries. The notations used in figures like FM and FSM represent the flexural mode and flexural symmetric mode, respectively.

6.1. Frequency. In Figs.1 and 2 the dispersion of frequency with wave number is studied for both thermally insulated and isothermal boundary of a circular plate immersed in fluid and in space. The first two longitudinal and flexural symmetric modes of vibration are

increases with increasing wave number for both thermally insulated and isothermal boundaries in Figs. 1 and 2. But there is small dispersion in the frequency in the current range of wave number in Fig. 3 for isothermal mode due to the combine effect of thermal relaxation and damping of fluid medium.

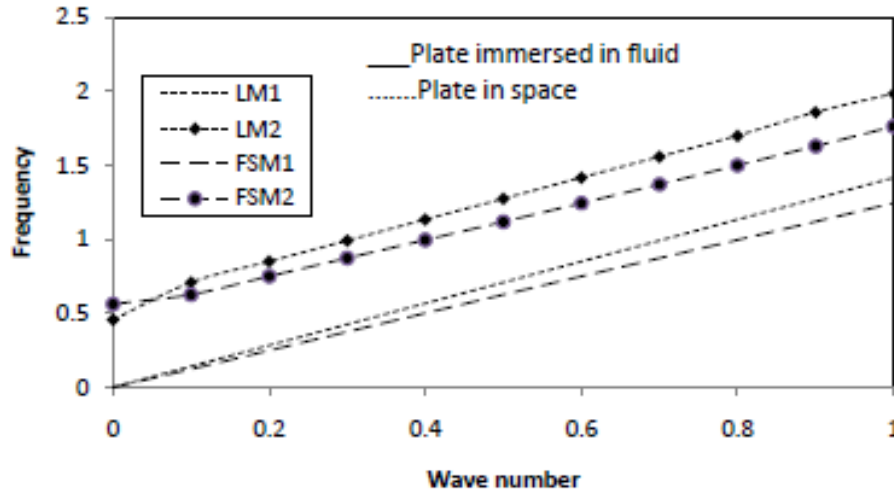


Fig. 1. Variation of non-dimensional frequency of thermally insulated circular plate with wave number.

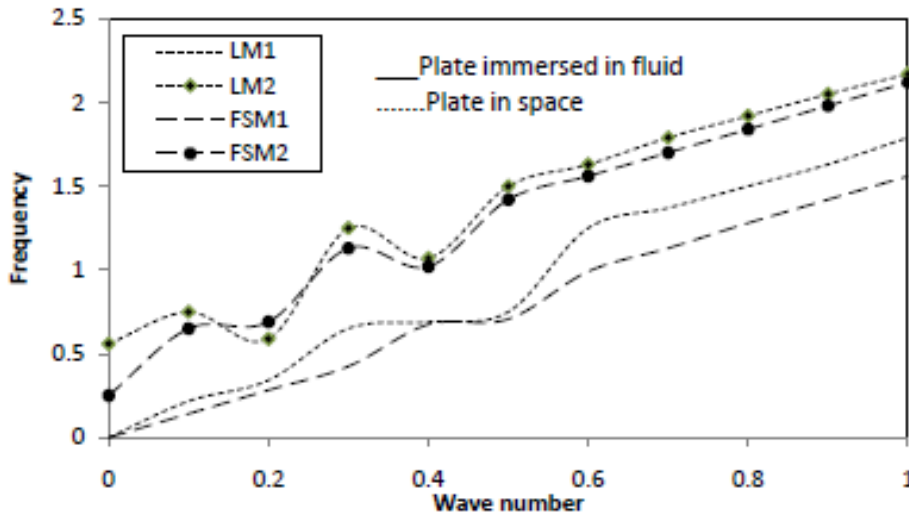


Fig. 2. Variation of non-dimensional frequency of isothermal circular plate with wave number.

6.2. Phase velocity. The variation of phase velocities with the wave number is discussed in Figs. 2 and 3 for both the thermally insulated and isothermal boundaries of the circular plate immersed in fluid and in space for the different modes of vibration. In Fig. 2 the phase velocity is decreasing at small wave number between 0 and 3 and become steady for higher values of the wave number for thermally insulated modes of vibration. For isothermal boundary there is a small deviation on the phase velocity in Fig. 3. The energy transmission occurs only on the surface of the plate because the plate acts as the semi infinite medium. The phase velocities of higher modes attain large values at vanishing wave number. From the Figs. 3 and 4 it is observed that the non-dimensional phase velocity of the fundamental mode

is non-dispersive and decreases rapidly in the presence of liquid and thermal relaxation times with increasing wave number.

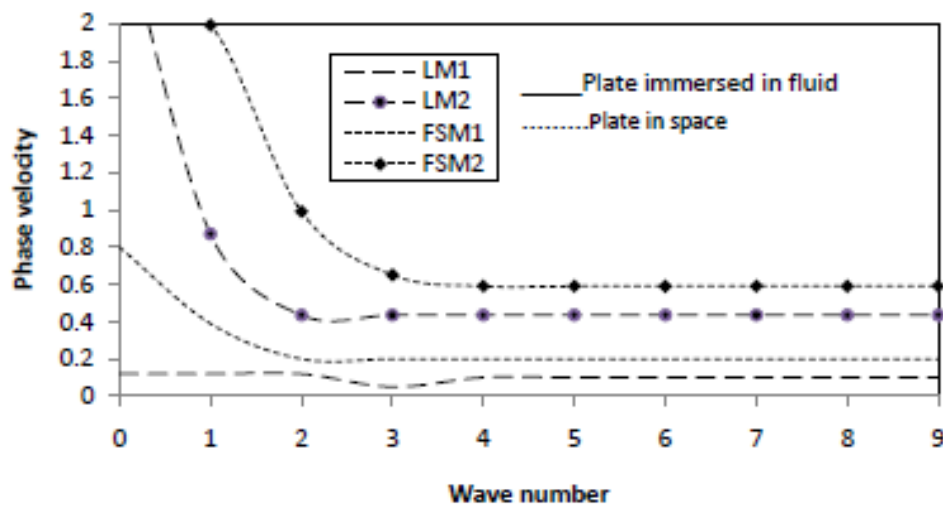


Fig. 3. Variation of non-dimensional phase velocity of thermally insulated circular plate with wave number.

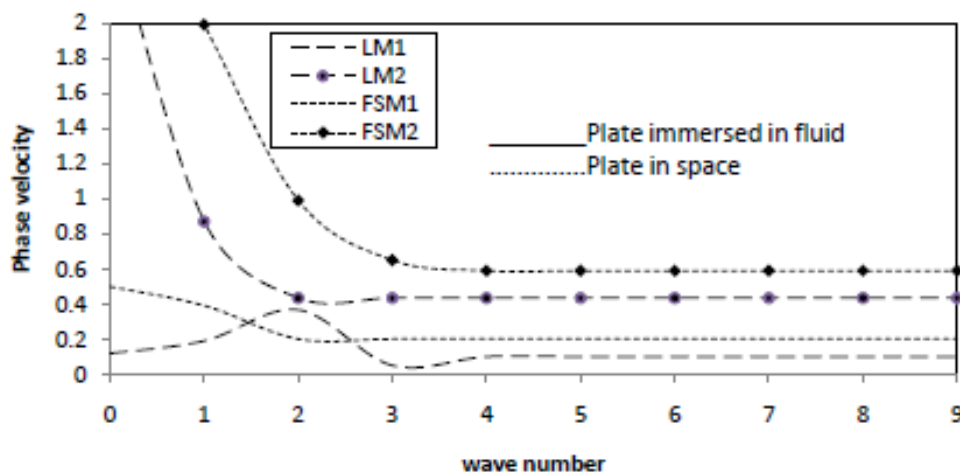


Fig. 4. Variation of non-dimensional phase velocity of isothermal circular plate with wave number.

6.3. Attenuation coefficient. In Fig. 5 the variation of attenuation coefficient with respect to wave number of circular plate with and without fluid is discussed for thermally insulated boundary. The magnitude of the attenuation coefficient increases monotonically to attain maximum value between 0.1 and 0.3 for first two modes of longitudinal and flexural (symmetric) vibration and slashes down to become asymptotically linear in the remaining range of wave number. The variation of attenuation coefficient with respect to wave number of isothermal circular plate is discussed in Fig. 6, here the attenuation coefficient attain maximum value in 0.1 and 0.4 with a small oscillation in the starting wave number and decreases to become linear due relaxation times. From Figs. 5 and 6, it is clear that the effects of stress free thermally insulated and isothermal boundaries of the plate are quite pertinent due to the combine effect of thermal relaxation times and damping effect of fluid medium.

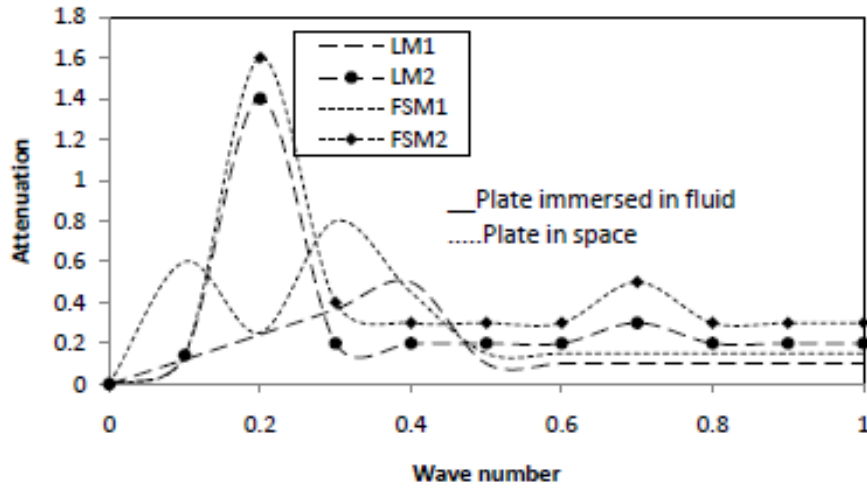


Fig. 5. Variation of attenuation of thermally insulated circular plate with wave number.

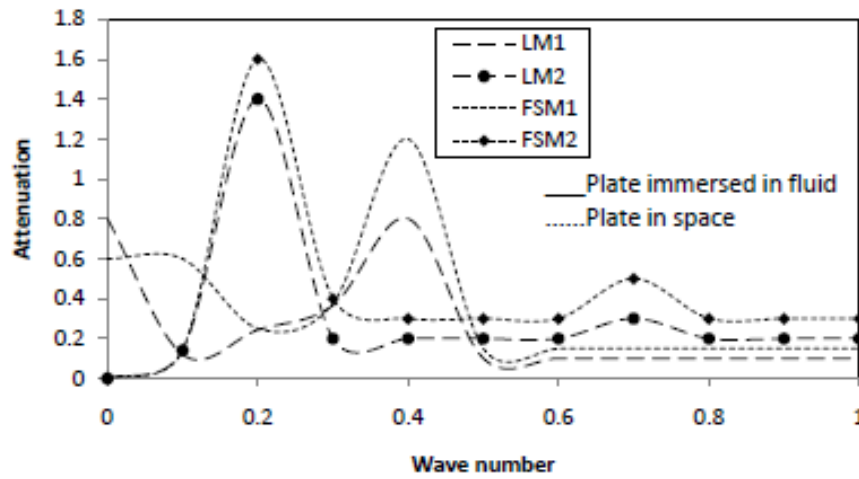


Fig. 6. Variation of attenuation of isothermal circular plate with wave number.

6.4. Relative frequency shift. The frequency shift of the wave due to rotation is defined as $\Delta\omega = \omega(\Omega) - \omega(0)$. Ω being the angular rotation; the relative frequency shift is given by

$$R.F.S = \left| \frac{\Delta\omega}{\omega} \right| = \left| \frac{\omega(\Omega) - \omega(0)}{\omega(0)} \right|, \quad (32)$$

where $\omega(0)$ is the frequency of the waves in the absence of rotation.

Relative frequency shift plays an important role in construction of gyroscope, acoustic sensors and actuators. Figs. 7 and 8 reveals that the variation of relative frequency shift with the wave number for first and second modes of longitudinal and flexural (symmetric) vibrations for thermally insulated and isothermal boundaries. The RFS is quite high at lower wave number and become diminishes with increasing wave number. The relative frequency shift profile are dispersive in the wave number range 0 and 0.4 and become steady in the remaining range of wave number for both thermally insulated and isothermal boundaries. The cross over points between the vibration modes represents the transfer of energy from solid to the fluid medium in all the dispersion curves.

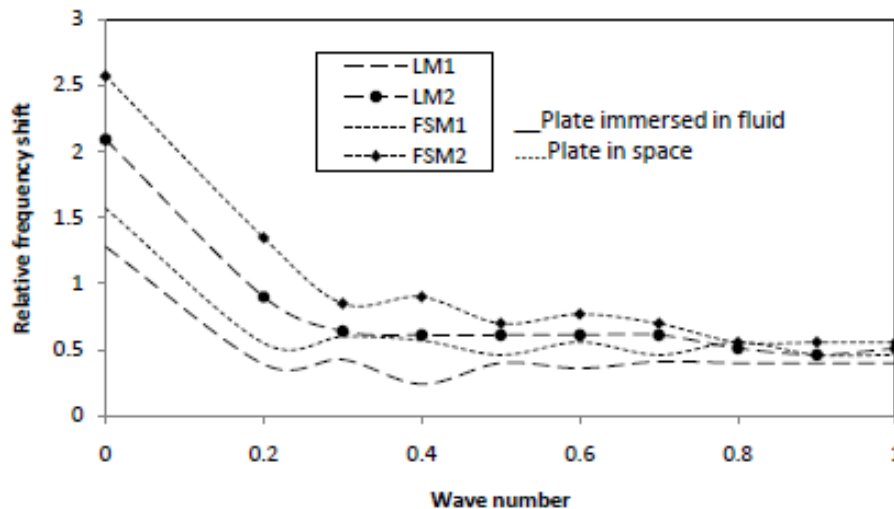


Fig. 7. Variation of relative frequency shift with wave number for thermally insulated circular plate.

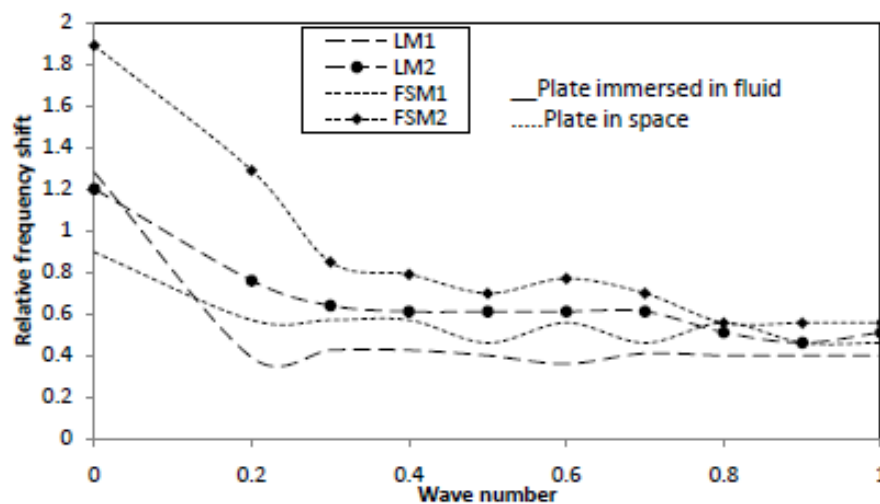


Fig. 8. Variation of relative frequency shift with wave number for isothermal circular plate.

7. Conclusions

The generalized thermoelastic waves in the rotating circular plate immersed in an inviscid fluid are studied based on the Lord-Shulman (LS) and Green-Lindsay (GL) generalized two dimensional theory of thermoelasticity. Two displacement potential functions are introduced to uncouple the equations of motion. The frequency equations that include the interaction between the plate and fluid are obtained by the traction free boundary conditions using the Bessel function solutions. The numerical calculations are carried out for the material Zinc and the computed non-dimensional frequency, phase velocity, attenuation coefficient and relative frequency shift are plotted as the dispersion curves for the plate with thermally insulated and isothermal boundaries. The wave characteristics are found to be more dispersive and realistic in the presence of thermal relaxation times, fluid and the rotation parameter.

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