CHAOTIC BREATHERS IN A NONLINEAR BEAM LATTICE MODEL

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Abstract. Dynamical behaviours of *chaotic breathers* are investigated numerically for a one-dimensional nonlinear beam lattice model involving rotational degree of freedom in addition to the well-known Fermi-Pasta-Ulam-β (FPU-β) lattice. Evolutions of the initial disturbances composed of the highest wavenumber mode are pursued numerically to observe generation, propagation, and an eventual decay of the localized structure (chaotic breather).

Detailed numerical analyses are done for the FPU-β lattice so as to compare with the beam lattice model. Initial localization process is discussed based on the modulational instability theory. Dynamical evolutions of displacemment, rotation, energy, wavenumber spectra etc. are observed and an attempt is made to understand qualitatively the entire evolution process. It is found that the inclusion of the rotation enhances either generation or decay processes of the chaotic breathers in comparison with the case of the FPU-β lattice.

1 Introduction

Recent investigations [1, 2, 3] have revealed a notion of the so-called “chaotc breather (CB)”, which is a sharp localized structure generated in numerical simulations for certain nonlinear lattices. The mode energy of the initial high-frequency π-mode is transferred through the modulational instability into a number of localized breather-like structures and then they merge into a few or a single localized structure with higher energy. They are found to be stable over relatively long time and each localized structure is qualitatively very similar to the discrete breather (DB) [4, 5] but behaves irregularly in its evolution. In some situations, CB finally breaks up and damps out into a kind of thermalized (equipartition) state. The most evident difference in exact DB and CB is that the latter moves in an erratic (chaotic) way and has mostly a finite lifetime.

Typical behaviours of CB have been observed for the FPU-β lattice and detailed investigations were done for this case [6, 7]. However, CBs are observed also for other lattice systems [8, 9]. The purpose of this paper is to investigate the properties of CB in a one-dimensional nonlinear beam lattice model described by the following nondimensionalized equations:

$$\begin{aligned} &\ddot{y}_n = (1+\alpha)(y_{n+1} - 2y_n + y_{n-1}) + \beta\{(y_{n+1}-y_n)^3 - (y_n - y_{n-1})^3\} - \frac{1}{2}(\psi_{n+1} - \psi_{n-1}), \\ &\psi_n = \frac{1}{2\gamma}(y_{n+1} - y_{n-1}) - \frac{1}{6\gamma}(\psi_{n+1} + 4\psi_n + \psi_{n-1}), \qquad (n = 1, 2, \ldots, N), \end{aligned} \tag{1}$$

where a dot denotes the time derivative, y_n and ψ_n denote displacement and rotation and $\alpha,\ \beta,$ and γ are constant parameters representing the measures of shear rigidity, nonlinearity, and rotation. This model is a nonlinear version of Askar’s model for molecular crystals [10, 11] and a simplified version of 1D crystals with complex lattice investigated by Potapov et al. [12]. In the case without rotation, this equation reduces to the FPU-β lattice.

This paper is organized as follows. In Section 2, method of numerical simulation and definition of notations utilized in this paper are provided. In Section 3, results of numerical simulations are shown for the FPU-β lattice. In Section 4, numerical consequences for the 1D beam lattice are discussed in comparison with the FPU-β case. Concluding remarks are given in the final section.

2 Numerical simulation and notation

The model equation (1) was solved by means of the 4th order Runge-Kutta method under the periodic boundary condition assigning the high-frequency π-mode initial condition. Namely the boundary condition is given by

$$y_0 = y_N, \quad \dot{y}_0 = \dot{y}_N, \quad y_{N+1} = y_1, \quad \dot{y}_{N+1} = \dot{y}_1,$$

$$\psi_0 = \psi_N, \quad \dot{\psi}_0 = \dot{\psi}_N, \quad \psi_{N+1} = \psi_1, \quad \dot{\psi}_{N+1} = \dot{\psi}_1.$$

Whereas the initial condition, i.e., the π-mode is given by

$$y_n = (-1)^n A_0, \quad \dot{y}_n = O(10^{-14}), \quad \psi_n = 0, \quad \dot{\psi}_n = 0.$$

Note here that very small perturbations were added to $\dot{y}_n$, otherwisw no modulation takes place because the π-mode is itself an exact solution to the FPU-β lattice.

Temporal evolutions of the displacement y_n and the rotation ψ_n were obtained by numerical integrations of (1) for the cases without and with the rotation and for various values of the lattice number (N), initial amplitude (A_0), and parameter values ($\alpha,\ \beta,\ \gamma$). To understand the results, evolutions of the following quantities are also considered.

From the Hamiltonian of Eq.(1), the energy associated with n-th particle is given by

$$E_n = \frac{1}{2}\dot{y}_n^2 + \frac{1}{2}\gamma\dot{\psi}_n^2 + \frac{1}{2}(1+\alpha)(y_n - y_{n+1})^2 + \frac{1}{4}(y_n - y_{n+1})^4 + \frac{1}{6}(\psi_n^2 + \psi_n\psi_{n+1} + \psi_{n+1}^2) + \frac{1}{2}(y_n - y_{n+1})(\psi_n + \psi_{n+1}). \quad (2)$$

For the later discussion, the energy E_n is divided into three parts associated with the displacement, the rotation, and the coupling of them, that is

$$E_n^{disp} \equiv \frac{1}{2}\dot{y}_n^2 + \frac{1}{2}(1+\alpha)(y_n - y_{n+1})^2 + \frac{1}{4}\beta(y_n - y_{n+1})^4,$$
$$E_n^{rot} \equiv \frac{1}{2}\gamma\dot{\psi}_n^2 + \frac{1}{6}(\psi_n^2 + \psi_n\psi_{n+1} + \psi_{n+1}^2),$$
$$E_n^{coup} \equiv \frac{1}{2}(y_n - y_{n+1})(\psi_n + \psi_{n+1}).$$

The total energy of the system is defined by taking the sum over the whole particle as follows: total energy $E_{total} \equiv \sum_{n=1}^{N} E_n$; correspondingly, *total displacement energy* $E_{total}^{disp} \equiv \sum_{n=1}^{N} {E_n}^{disp}$; *total rotation energy* $E_{total}^{rot} \equiv \sum_{n-1}^{N} {E_n}^{rot}$; *total coupling energy* $E_{total}^{coup} \equiv \sum_{n=1}^{N} {E_n}^{coup}$. We introduce the localized energy (or the peak energy) E_{pk} by

$$E_{pk} \equiv E_{i-1} + E_i + E_{i+1}, \qquad E_i \equiv max[E_n],$$

that is, the energy of a localized structure is defined by the sum of three particles' energies around the peak. Furthermore, *maximum localized energy* $max[E_{pk}]$ is defined by the maximum value of E_{pk} in the whole lattice. To quantify the strength of localization of chaotic breather, *ratio of localization* R_{CB} is introduced by

$$R_{CB} \equiv max[E_{pk}] / \sum_{n=1}^{N} E_n.$$

In addition to the displacement and the rotation, we also consider the Fourier transform of them. Then the frequency spectrum for displacement y_n is denoted by Y_k and that for rotation ψ_n is by Ψ_k with the wavenumber k.

3 Results for FPU-β lattice

In case of the FPU-β lattice, the following scenario of evolutions has been reported, for instance, in [2]. The initial π-mode becomes modulationally unstable when its energy exceeds a threshold and develops

into a number of discrete breathers. They merge into a single localized structure (single CB) through interactions and it propagates in the lattice in an erratic way. This stage is termed as *localized state*. The single CB then decays into a kind of thermalized state, which is termed as *equipartition state*.

Numerical simulations were carried out for a variety of different values of the particle number N and the initial amplitude A_0 of the π-mode under the boundary and the initial conditions explained in the previous section. Some results for the FPU-β lattice, which follow essentially the above-mentioned scenario, are shown in the sequel.

An example of typical stages of evolutions of $y_n,\ Y_k$ and E_n are shown in Fig.1 for the case with $\alpha = 0, \beta = 0.1, N = 64, A_0 = 1.0$. Three stages are shown in Fig.1, that is, (a) modulated state ($t = 169$), (b) localized state ($t = 4000$) and (c) equipartition state ($t = 8000$). Panel (a) shows a modulated state where 9 DBs arise due to the modulational instability of the initial π-mode. These 9 DBs interact with each other and merge into one big DB at around $t = 2500$ and it moves around the lattice until around $t = 5500$. Panel (b) at $t = 4000$ shows such a localized state (i.e., CB). Then at around $t = 6000$ this localized structure decays into an equipartition state. Panel (c) represents a typical pattern in such a state where the energy is transferred to lower wavenumber components.

Figure 2 shows variations of the localized energy E_{pk} for different values of the particle number N with $A_0 = 1.0$ and $A_0 = 1.5$. This figure shows that E_{pk} increases as N increases and the decay time $t_d \sim 6000$ for $A_0 = 1.0$ or $t_d \sim 1200$ for $A_0 = 1.5$ is independent of N. Similar calculations were done for $A_0 = 0.5$ and $A_0 = 2.0$. These results indicate that E_{pk} increases as A_0 increases while t_d decreases as A_0 increases.

Figure 1: Evolutions of $y_n,\ Y_k,$ and E_n for the FPU-β lattice ($\alpha = 0,\ \beta = 0.1,\ N = 64,\ A_0 = 1.0$). (a) modulated state ($t = 169$), (b) localized state ($t = 4000$), (c) equipartition state ($t = 8000$).

The criterion for the modulational instability of the initial π-mode can be derived as was given in [13, 14, 15] and we have the dispersion relation

$$[(\Omega+\omega)^2 - 4(1+2\delta)\sin^2(\frac{q+Q}{2})][(\Omega-\omega)^2 - 4(1+2\delta)\sin^2(\frac{q-Q}{2})] = 4\delta^2(\cos Q - \cos q)^2, \quad \delta \equiv 3\beta A_0^2 \sin^2(q/2), \tag{3}$$

where q and ω are wavenumber and frequency of the initial plane wave and Q and Ω are those of the prturbation. The growth rate $\sigma(\pi, Q)$ of the modulational instability for the π-mode is given by

$$\sigma(\pi, Q) = 2\{[4(1+\delta)(1+2\delta)\cos^2(Q/2) + \delta^2\cos^4(Q/2)]^{\frac{1}{2}} - [1+\delta+(1+2\delta)\cos^2(Q/2)]\}^{\frac{1}{2}}.$$

Figure 3 shows the growth rate $\sigma(\pi, Q)$ for $N = 64$, which shows that it increases as A_0 increases. In numerical simulations, the critical enery E_c for the modulational instability was examined by changing the particle number N and the nonlinearity parameter β, and it was observed that E_c decreases as N and β increase, giving rise to a relation $E_c \propto 1/\beta N$. Furthermore, by comparing with numerical results, it is found that the first growing mode in simulations is $Q_f (Q_{max} < Q_f < \pi)$ but not the maximum growing mode Q_{max} of σ. The wavenumber of the modulation observed in numerical simulations is $\pi - Q_f$.

Evolutions of the Fourier spectra Y_k and Ψ_k were exhibited in color gradation to observe detailed energy distributions in the wavenumber space. However, only a 3D plot of Y_k for the FPU-β in the case with $N = 32, A_0 = 1.0$ is shown here in Fig.4 as an example of the spectral evolution. It can be seen that the maximum spectrum is initially at the highest wavenumber but the energy is transferred to the lowest wavenumber in the equipartition state.

The maximum localized energy $max[E_{pk}]$ and the ratio of localization R_{CB} are shown in Fig.5(a) and (b) for $A_0 = 1.0, 1.5$ and 2.0. It can be observed that $max[E_{pk}]$ increases as A_0 and N increase but it saturates as N tends to larger value. The ratio of localization R_{CB} decreases as N increases, since $max[E_{pk}]$ saturates as N increases.

Figure 2: Variation of localized energy E_{pk} for various N with (a) $A_0 = 1.0$ and (b) $A_0 = 1.5$

Figure 3: Growth rate σ for the modulational instability of π-mode ($N = 64$)

Figure 4: 3D plot of evolutions of wavenumber spectrum Y_k for $N = 32$ and $A_0 = 1.0$

Figure 5: The maximum energy (a) $max[E_{pk}]$ and the ratio of localization (b) R_{CB} versus N

4 Resuts for 1D beam lattice model

The effect of rotation is included now. Equation (1) including ψ_n is solved numerically under the boundary and the initial conditions given in Section 2. Although the scenario of evolutions is similar to the case of the FPU-β lattice, generally speaking, much faster decay of the localized state into the equipartition state is observed for the beam lattice case.

Variations of E_{pk} (averaged over smples of different initial perturbations on $\dot{y}_n$) are shown in Fig.6 for different values of A_0 with $\gamma = 1.0$. The decay time t_d is given roughly by $t_d \sim 2300$ for $A_0 = 1.0$, $t_d \sim 700$ for $A_0 = 1.5$ and $t_d \sim 350$ for $A_0 = 2.0$, thus t_d is smaller than that for the corresponding case in the FPU-β lattice.

Figure 7 shows evolutions of the localized energy E_{pk} for different values of the rotation parameter γ when $N = 32$ and $A_0 = 1.0$. No localization occurs for $\gamma = 0.2$ but the modulational instability occurs at almost the same time when $\gamma \geq 0.5$. The value of E_{pk} becomes larger as γ increases and the decay time t_d becomes shorter as γ increases. For $\gamma = 5.0$, the localized state exists until around $t \sim 1200$ but very quickly decays into equipartition state. Thus either the localization or the decay into equipartition are enhanced due to the effect of rotation.

Variations of the three components of the total energy, E^{disp}_{total}, E^{rot}_{total} and E^{coup}_{total}, are shown in Fig.8. This figure provides the following properties. In the localized state ($t \leq 1500$ for $\gamma = 1.0$, $t \leq 1200$ for $\gamma = 5.0$), three components of energy take almost constants values, but in the equipartition state (after the decay of CB), E^{disp}_{total} decreases and E^{rot}_{total} increases, while E^{coup}_{total} takes negative values. Then a part of the displacement energy is transferred to the rotation energy and they are almost equibrated as $E^{disp}_{total} \approx E^{rot}_{total}$.

In terms of the normal mode analysis, we can show, for the 1D beam lattice, that the coupling energy becomes negative for the acoustic type mode and positive for the optical type mode [16]. Therefore, we may pose a conjecture that the numerical result $E^{coup}_{total} < 0$ implies an excitation of the acoustic type waves in the equipartition state. It is also probable that this conjecture may also be applicable to the FPU-β case, that is, the energy of the chaotic breather transfers to that of the usual solitary wave type low wavenumber modes in the equipartition state. This point should be explored in more details.

Figure 6: Variations of localized energy E_{pk} for different N with $\gamma = 1.0$: (a) $A_0 = 1.0$, (b) $A_0 = 1.5$, (c) $A_0 = 2.0$

5 Concluding remarks

We have investigated the evolution of the initial π-mode for the 1D beam lattice model in comparison with the FPU-β lattice. Consequences obtained so far can be summarised as follows:

1) The growth rate (σ) of the modulational instability increases as the initial amplitude (A_0) increases. The critical energy (E_c) for instability decreases as nonlinearity (β) or particle number (N) increases and gives a relation $E_c \propto 1/\beta N$.

2) Wavenumbers of the modulation found in numerical results are $\pi - Q_f$ but not Q_{max}. Here Q_f is the first growing mode in numerical simulation ($Q_{max} < Q_f < \pi$) and Q_{max} is the maximum growing mode of σ given by the dispersion relation (Eq. (3)) for the modulational instability of the $\pi-$mode.

3) The localized energy (E_{pk}) increases as N and A_0 increase, and $max[E_{pk}]$ is proportional to the total energy, that is, $max[E_{pk}] \propto NA_0^2$. The decay time ($t_d$), until then the localization (CB) survives, decreases as A_0 increases independent of N, and thus t_d is found to be dependent on the initial energy density (A_0^2).

4) The localized energy (E_{pk}) tends to a constant as $N \to \infty$, while the ratio of localization (R_{CB}) decreases as $N \to \infty$, which indicates the existence of the upper limit for the localization (E_{pk}) as N increases.

Figure 7: Variations of localized energy E_{pk} for different values of γ ($N = 32,\ A_0 = 1.0$)

Figure 8: Variations of three components (displacement, rotation, and coupling) of total energy for different values of γ ($N = 32,\ A_0 = 1.0$): (a) E^{disp}_{total}, (b) E^{rot}_{total}, (c) E^{coup}_{total}

5) Inclusion of the rotation (ψ_n) enhances the localization (CB) and leads to a quick decay into the equipartition. The decay time (t_d) decreases as the rotation parameter γ increases. Rapid decay of E_{pk} into the equipartition and rapid increase of the rotation energy E^{rot}_{total} are observed for the case with the rotation.

6) In the equipartition state, three components of the total energy satisfy $E^{disp}_{total} \approx E^{rot}_{total}$ and $E^{coup}_{total} < 0$. The negativity of the coupling energy implies a possibility of excitation of the acoustic type modes in the equipartition state.

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