NONLINEAR WAVES IN 1-D SOLIDS WITH MICROSTRUCTURE

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Abstract. A general model of one-dimensional body with a scalar microstructure is introduced. Field equations are obtained via a variational principle, as Euler-Lagrange equations of a suitable energetic functional. The evolution of finite amplitude strain solitary waves is studied, taking into account both micro and macro dissipations. The formation, propagation and attenuation/amplification of bell-shaped and kink-shaped waves is proved. For a very simple form of the modal equation, the nonlinearity in the microlevel leads to a complicated term in the equation of motion and opens up direct ways for determining material constants characterizing the microstructure.

1 Introduction

The presence of a microstructure in an elastic solid may have both dispersive and dissipative effects. The dynamical excitations have the same wavelengths as the spatial scale of inhomogeneities. There is a large literature on the applications of the classical theory of microstructures to wave propagation, see [8, 18] (and references therein), both in the linear and non linear cases [8, 11]. Contemporary understanding of basic mechanics of such materials is presented in Maugin [15]. In the usual models for nonlinear strain waves in elastic solids, the influence of macrodissipation may be described by various methods, see [6] and references therein. Recently, in a series of papers [4, 9], the simultaneous influence of macro and microdissipation on shock-wave propagation has been discussed, neglecting the inertia of the microstructure and microstresses possibly associated with the fine structure. Later numerical simulations were performed to study for the evolution of periodic waves [13].

In the present paper the model presented in [4, 9] is modified, including both the inertia of the microstructure and the microstress. As a result, a nonlinear PDE , with dispersion and dissipative (active) terms, is derived for one-dimensional bodies with scalar microstructures. More general equations for the three-dimensional bodies, which include this model as a particular case, have been obtained by Pastrone [20]. The one-dimensional equation admits as solutions both bell-shaped and kink-shaped longitudinal strain solitary waves. The simultaneous influence of the macro- and micro-dissipation and instability on the wave evolution is studied, and the formation of quasistationary dissipative solitary waves is discussed. The explanation of energy transmission by the microstructure, as well as the possibility of the estimation of the microstructure parameters, are proposed on the basis of the results obtained.

In the final Section we recall some results obtained by Engelbrecht [10], following the idea of the hierarchy in the sense of Whitam [25], namely to obtain a governing equation which should be able to distinguish between the scales. The concept of internal variables can be useful in deriving the evolution equations (see [9]). The evolution equation can be obtained as well, on the basis of a "skeleton" equation, which can be enriched by various constitutive assumptions physically meaningful, as shown in [10].

2 The field equations for dissipative 1-D microstrucure

Let $\mathcal{B}$ a 1-D body and $\mathbf{u} = u\,\mathbf{e}$ the position vector, where $\mathbf{e}$ is a unit vector of E_1, $u = u(x,t)$ represents the displacement field, x is a lagrangian coordinate, t is time. The microfield is characterized by the vector field $\mathbf{d} = \psi\,\mathbf{e}$, and $\psi = \psi(x,t)$ describes the microstructure (i.e., a density of microcracks, of defects, etc.). The fields are supposed to be smooth enough. We assume that the constitutive functions can be derived from a strain energy density in the form

$$W = W(u, u_x, \psi, \psi_x, x). \tag{1}$$

The kinetic energy is defined as a quadratic function of u_t and d_t and, for brevity's sake, we can diagonalize it

$$T = \frac{1}{2}(\rho u_t^2 + I\psi_t^2). \tag{2}$$

In (1) and (2) we denote with a subscript the partial derivative with respect to the corresponding variable, ρ is 1-D mass density, I an inertia term.
The general field equations can be derived, via a variational principle, from the energy functional:

$$\mathcal{E} = \int \rho(T + W)\,\mathrm{d}x. \tag{3}$$

The proper Euler-Lagrange equations are

$$\begin{aligned} &\left(\frac{\partial W}{\partial u_x}\right)_x - \frac{\partial W}{\partial u} = \rho u_{tt} \\ &\left(\frac{\partial W}{\partial \psi_x}\right)_x - \frac{\partial W}{\partial \psi} = I\psi_{tt} \end{aligned} \tag{4}$$

If we set

$$\begin{aligned} \sigma_{\mathrm{eq}} &= \frac{\partial W}{\partial u_x}\,, \quad b_{\mathrm{ext}} = -\frac{\partial W}{\partial u} \\ \tau_{\mathrm{eq}} &= -\frac{\partial W}{\partial \psi}, \qquad \eta_{\mathrm{eq}} = \frac{\partial W}{\partial \psi_x} \end{aligned} \tag{5}$$

we get the field equations obtained in [8, 9, 23] from integral balance equations, in various form. The meaning of the left-side terms in (5) is: σ_{eq} macrostress, τ_{eq} internal microforce, η_{eq} microstress (including micro-macro interaction).
We neglect the external body forces and the field equations take the form:

$$\rho\, u_{tt} = \sigma_x, \qquad I\psi_{tt} = \tau + \eta_x. \tag{6}$$

This model does not take into account dissipation, which can be introduced through the total (locally) power expended

$$P_T = \frac{\mathrm{d}W}{\mathrm{d}t} + \mathcal{D}, \tag{7}$$

where $\mathcal{D}$ is the dissipation and must satisfy the inequality $\mathcal{D} \geq 0$, for any admissible deformation. In our case $\mathcal{D}$ takes the form:

$$\mathcal{D} = \sigma_{\mathrm{neq}} u_{xt} + \tau_{\mathrm{neq}} \psi_t + \eta_{\mathrm{neq}} \psi_{xt} \geq 0, \tag{8}$$

and σ_{neq}, τ_{neq}, η_{neq} are the non-equilibrium (dissipative) fields. Hence, the stresses can be written as

$$\sigma = \frac{\partial W}{\partial u_x} + \sigma_{\mathrm{neq}}\,, \quad \tau = -\frac{\partial W}{\partial \psi} - \tau_{\mathrm{neq}}\,, \quad \eta = \frac{\partial W}{\partial \psi_x} + \eta_{\mathrm{neq}} \tag{9}$$

and the field equations (6) become

$$\rho u_{tt} = \sigma_x = \left(\frac{\partial W}{\partial u_x}\right)_x + (\sigma_{\mathrm{neq}})_x \tag{10}$$

$$I\psi_{tt} = \tau + \eta_x = \left(\frac{\partial W}{\partial \psi_x}\right)_x + (\eta_{\mathrm{neq}})_x - \frac{\partial W}{\partial \psi} - \psi. \tag{11}$$

In principle the dissipative stresses depend on the whole set of variables $(u_x,\, \psi,\, \psi_x,\, u_{xt},\, \psi_t,\, \psi_{xt})$, but we can simplify the model and obtain meaningful results by assuming that the nonequilibrium fi elds depend linearly on the rate variables $(u_{xt}, \psi_t, \psi_{xt})$, are independent of (u_x, ψ) and have the form:

$$\begin{aligned} \sigma_{\text{neq}} &= D_{11} u_{xt} + D_{12} \psi_t, \\ \tau_{\text{neq}} &= D_{21} u_{xt} + D_{22} \psi_t, \\ \eta_{\text{neq}} &= 0. \end{aligned} \tag{12}$$

We also assume that W is quadratic in the ψ and ψ_x, and account for nonlinearity in u_x through third-degree terms in u_x (Murnaghan model):

$$W = \frac{1}{2}\alpha u_x^2 + \frac{1}{3}\beta u_x^3 - A\psi u_x + \frac{1}{2}B\psi^2 + \frac{1}{2}C\psi_x^2, \tag{13}$$

with α, β, A, B, C assigned constants related to the usual elastic moduli as shown in [22].

The fi eld equations may be further simplifi ed by considering only long waves with characteristic length $L >> 1$. We also assume that the typical size of the strain is small, say $V << 1$. Moreover, the microinertia term I may be assumed to depend upon the square of a typical size l of a microstructure element, and I may be rewritten as $I = \rho l^2 I^*$, with I^* dimensionless. Using dimensional analysis, one can assume that $C = l^2 C^*$ with C^* having the dimension of stress. Finally, notice that the influence of dissipation may be conveniently described by assuming that $D_{ij} = dD^*_{ij}$, where d has the dimension of length.

Consistently, we introduce the dimensionless parameters

(a) $\varepsilon := V << 1$, which accounts for the elastic strains;

(b) $\delta := l^2/L^2 << 1$, which characterizes the ratio between the microstructure size and the wave length;

(c) $\gamma := d/L$, characterizing the influence of the dissipation.

Consider now a rescaling of the space variable x by L, of the strain v and the microstructural variable ψ by V, and of the time variable t by L/c_0, with $c_0^2 = (\lambda + 2\mu)/\rho$ a characteristic velocity.

The evolution of nonlinear strain waves depends upon the ratio between the parameters ε, γ and δ. This the reason we retain terms quadratic in these parameters in the expansion of the solution (14). Sometimes they can be considered "negligibly small", sometimes not, according to the different effects we want to point out, as it will be explained in the following Sections. In this paper we shall neglect the terms multiplying γ^2 and δ^2 in the following development, so that the model equation for subsequent development is

$$\begin{aligned} v_{tt} - v_{xx} - \varepsilon\alpha_1 (v^2)_{xx} - \gamma\alpha_2\, v_{xxt} + \delta(\alpha_3\, v_{xxxx} - \alpha_4\, v_{xxtt}) + \\ + \gamma\delta(\alpha_5\, v_{xxxxt} + \alpha_6\, v_{xxttt}) = 0, \end{aligned} \tag{14}$$

which happens to be the same equation obtained in a more phenomenological way in [21].

3 Bell-shaped solitary waves

The balance between nonlinearity and dispersion takes place when $\varepsilon = O(\delta)$. If $\gamma = 0$ we have nondissipative case governed by the double dispersive equation,

$$v_{tt} - v_{xx} - \varepsilon(\alpha_1 v_{xx}^2 - \alpha_3\, v_{xxxx} + \alpha_4\, v_{xxtt}) = 0. \tag{15}$$

Its exact bell-shaped travelling solitary wave solution arises as a result of balance between nonlinear and dispersive terms. It satisfi es the boundary conditions

$$\frac{\partial^k}{\partial x^k} v \to 0 \text{ at } |x| \to \infty,\ k = 0, 1, 2, 3, \tag{16}$$

and has the form

$$v = \frac{6k^2(\alpha_4\, c^2 - \alpha_3\,)}{\alpha_1} \cosh^{-2}(k\,(x - ct)), \tag{17}$$

where c is a free parameter,

$$k^2 = \frac{c^2 - 1}{4\varepsilon(\alpha_4 c^2 - \alpha_3)}. \tag{18}$$

Hence, solitary wave (17) exists when $c^2 > \max\{1, \alpha_3/\alpha_4\}$ or when $0 < c^2 < \min\{1, \alpha_3/\alpha_4\}$. In first case longitudinal tensile waves propagate while in second case only compression waves propagation occurs.

As shown in [21], in general Eq.(14) admits an exact solution of the type "travelling bell shaped", but such solution exists only under particular values of the equation coefficients. Exact bell-shaped solution appears as a result of simultaneous balance between higher order dissipative terms, and nonlinearity and dispersion.

When $\varepsilon = O(\delta)$, $\gamma = \varepsilon\gamma_1$, $\gamma_1 << 1$, the dissipation (or accumulation) $\alpha_2\, v_{xxt}$ acts as a perturbation to the balance between nonlinearity and dispersion while higher order dissipative terms may be neglected. Asymptotic solution of the Eq.(14) predicts either the infinite growth finite time for solitary wave (17) or its vanishing at $t \to \infty$.
More interesting scenario of the solitary wave evolution may be studied considering Eq.(14) as dissipation perturbed double dispersive equation (15) with $\gamma << 1$. The asymptotic solution is sought as a function of the phase variable $\theta = x - ct$ and the slow time T, $v = v(\theta, T)$, with

$$\theta_x = 1,\ \theta_t = -c(T),\ T = \gamma\, t.$$

Then from (14) we get

$$(c^2-1)v_\theta - \varepsilon\alpha_1(\,v^2)_\theta + \varepsilon(\alpha_3 - \alpha_4 c^2)\, v_{\theta\theta\theta} = \gamma\Big(2c[v_T - \varepsilon\alpha_4\, v_{\theta\theta T}] + \\ + c_T[v - \varepsilon\alpha_4\, v_{\theta\theta}] - c\frac{\partial^2}{\partial\theta^2}\left[\alpha_2\, v - \varepsilon(\alpha_5 + \alpha_6 c^2)\, v_{\theta\theta}\right]\Big) + O(\gamma^2). \quad (19)$$

The solution of Eq.(19) is sought in the form

$$v = v_0(\theta, T) + \gamma v_1(\theta, T) + ... \quad (20)$$

where v_i, $i = 0, 1, ...$, satisfy boundary conditions (16) at $|\theta| \to \infty$. Substituting (20) into (19) we have in the leading order

$$(c^2 - \alpha_1)v_{0,\theta} - \varepsilon\alpha_1(\,v_0^2)_\theta + \varepsilon(\alpha_3 - \alpha_4 c^2)\, v_{0,\theta\theta\theta} = 0. \quad (21)$$

The exact solitary wave solution of Eq.(21) has the form (17). Then using (21) and boundary conditions at infinity one can obtain the equation for the function $s = c^2$,

$$7\varepsilon\, s_T\; Q_3(s) = 2s\,(s - \alpha_1)^2(q_1\, s^2 + q_2\, s + q_3), \quad (22)$$

where $Q_3(s)$ is a third degree polynomial in s, q_1, q_2, q_3 are suitable combinations of α_i (see [21]).

Important features of the behavior of s may be established analyzing (22) without integration. The most interesting evolution of s is realized when s (and hence the amplitude of the solitary wave (17)) tends to the finite constant value s^* as $T \to \infty$. We shall call it the "selection" of the solitary wave. The values of s^* are the real positive solutions of equation

$$q_1\, s^2 + q_2\, s + q_3 = 0. \quad (23)$$

Inspection of the plot of the solution shows that the solitary wave keeps its localized undisturbed bell-shaped form (see [22]). On amplification besides growth of the wave amplitude the width of the wave decreases. The attenuation is provided by simultaneous decrease of the amplitude and the increase of the width. The addition of correction γv_1 doesn't change significantly profile of $v = v_0$. More important is that the asymptotic solution (20) is not uniformly valid. The matching asymptotic expansions method should be applied to complete the solution, it also doesn't alter the behavior of the wave near the core. A more detailed discussion can be found in [21, 22].

4 Kink-shaped solitary waves

Eq.(14) possesses variety of exact travelling kink-shaped solutions [19], in the form of

$$v \;=\; \sum_{k=0}^{3} A_k\, \tanh^k(m\,\theta), \quad (24)$$

but all of them have fixed wave numbers, and the additional restrictions on the equation coefficients are needed.

When dispersion is weak $\delta < \varepsilon$ and $\gamma = O(\varepsilon)$ nonlinearity is balanced by the dissipation only. Then Eq.(14) may be rewritten as

$$v_{tt} - v_{xx} - \varepsilon(\alpha_1 v_{xx}^2 + \alpha_2 v_{xxt}) = \\ = -\delta(\alpha_3\, v_{xxxx} - \alpha_4\, v_{xxtt}) - \varepsilon\delta(\alpha_5\, v_{xxxxt} + \alpha_6\, v_{xxttt}), \tag{25}$$

whose solution is sought in the form

$$v = v_0 + \delta v_1 + ... \tag{26}$$

where $v_i = v_i(\theta = x - ct)$ satisfy the boundary conditions

$$v_0 \to h_\pm,\; v_i \to 0,\; i > 0,\; \text{for } \theta \to \pm\infty, \tag{27}$$

and with all derivatives of v_i with respect to θ vanishing at infinity. For a kink $h_+ \neq h_-$. In the leading order the kink solution has the form

$$v \;=\; A\, m\, \tanh(m\,\theta) + B, \tag{28}$$

with

$$A = -\frac{\alpha_2 c}{\alpha_1},\;\; B = \frac{c^2 - 1}{2\varepsilon\alpha_1}. \tag{29}$$

There are two free parameters, the phase velocity c and the wave number m, whose explicit value is given in [21, 22].

The solutions obtained may arise when the initial condition has the form of undisturbed kink (28) that has been shown numerically in [24].

Case $\delta = O(\varepsilon)$, $\gamma = O(\varepsilon)$ corresponds to the simultaneous balance between nonlinearity, dispersion and dissipation, perturbed by the higher order dissipative terms,

$$v_{tt} - v_{xx} - \varepsilon[\alpha_1 v_{xx}^2 + \alpha_2 v_{xxt} - \alpha_3 v_{xxxx} + \alpha_4 v_{xxtt}] = -\varepsilon^2(\alpha_5\, v_{xxxxt} + \alpha_6\, v_{xxttt}), \tag{30}$$

The solution of Eq.(30) is sought in the form

$$v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 ... \tag{31}$$

where boundary condition (27) hold. Substituting (20) into (30) in the leading order we have the wave equation having D'Alembert solution. Consider only one wave travelling to the left and assume v depends upon phase variable $\theta = x - ct$ with $c = 1 + \varepsilon c_1 + \varepsilon^2 c_2$.... Then at order ε the ODE equation for v_0 is,

$$2c_1 v_{0,\theta} - \alpha_1 (v_0^2)_\theta + \alpha_2 v_{0,\theta\theta} + (\alpha_3 - \alpha_4) v_{0,\theta\theta\theta} = 0. \tag{32}$$

The equation (32) possesses exact kink-shaped solution,

$$v_0 = A \tanh(m\,\theta)\, \text{sech}^2(m\,\theta) + B\;\tanh(m\,\theta) + C, \tag{33}$$

where a, B, C and m can be written in terms of α_i (see [21]).

It follows from (27) that

$$h_+ - h_- = 2B,\; c_1 = \frac{\alpha_1}{2}(h_+ + h_-),$$

and solution exists under special boundary conditions. The influence of higher order terms may be studied similar to the solution (26).

5 Nonlinearities in microscale and wave hierarchy

In this Section we focus our attention on another feature of the fi eld equations discussed above, namely on the possibility of introducing the idea of hierarchy of waves in the sense of Whitham [25].

This approach has been developed in [10] and here we recall just the main results. The main point of this approach is to construct a governing equation which should be able to distinguish between the scales (long- and short- wave excitations). The concept of internal variables can be extremely useful in deriving the evolution equation.

Let us proceed with the equations (**??**), (11) and, as a fi rst step, we analyse only the dissipationless case.

The simplest free energy function describing the influence of a microstructure is a quadratic function like (13), with $\beta = 0$.

The balance equations are now specifi ed as follows:

$$\rho\, u_{tt} = \alpha\, u_{xx} - A\, \psi_x, \tag{34}$$

$$I\, \psi_{tt} = C\, \psi_{xx} - A\, u_x + B\, \psi. \tag{35}$$

For further analysis we introduce dimensionless variables (note that ψ is already dimensionless)

$$U = u\, U_0^{-1}, \quad X = x\, L^{-1}, \quad T = t\, c_0\, L^{-1}, \tag{36}$$

where $c_0^2 = \alpha\, \rho^{-1}$; U_0 and L are certain constants (intensity and wavelenght of the initial excitation). We also need a scale of the microstructure *l*. Then two dimensionless parameters can be introduced: $\delta \approx l^2 L^{-2}, \epsilon \approx U_0 L^{-1}$. Following [porpastr-int.j.mat.mech], we suppose $I = \rho\, l^2\, I^*$ and $C = l^2\, C^*$, where I^* is dimensionless and C^* has the dimension of the stress.

Then (34),(35) yields

$$U_{TT} = U_{XX} - \frac{A}{\epsilon\, \rho_0 c_0^2} \psi_X, \tag{37}$$

$$\delta\, \alpha\, I^* \psi_{TT} = \delta\, C^* \psi_{XX} - A\, \epsilon\, U_X + B\, \psi. \tag{38}$$

In order to reduce the system (37), (38) into one governing equation, the slaving principle could be used [22]. Indeed (38) yields

$$\psi = \frac{\epsilon\, A}{B} U_X + \frac{\delta}{B} (\alpha\, I^* \psi_{TT} - C^* \psi_{XX}). \tag{39}$$

If we consider $\psi = \psi_0 + \delta\, \psi_1 + \ldots$, we get

$$\psi_0 = \epsilon\, A\, B^{-1} U_X, \tag{40}$$

$$\psi_1 = \epsilon\, \alpha\, A\, I^* B^{-2} U_{XTT} - \epsilon\, \alpha\, A\, C^* B^{-2} U_{XXX}. \tag{41}$$

With somw calculations, explicitely given in [10], we get the fi nal equation in terms of U as follows:

$$U_{TT} = \left(1 - \frac{A^2}{\alpha\, B}\right)\, U_{XX} + \delta \frac{A^2 I^*}{B^2} \left(U_{TT} - \frac{C^*}{\alpha\, I^*} U_{XX}\right)_{XX}. \tag{42}$$

This is sought “skeleton” of the wave equation for microstructured solids, much in the sense of the Lorenz system for the Navier-Stokes equations.

For a more detailed discussion of the properties of equation (42) we refer to [10]. We just remind that it reflects the wave hierarchy in Whitham sense [25], the effect of the microstrucure on the wave speed is given by *A*, and the influence of the microstructure is characterized by dispersive terms.

The model of the one-dimensional chain of dumbbell-like particles, exhibiting transverse displacements V_n and rotations ψ_n, yields a similar result [17]. Indeed, after passing to the continuum limit, the fi nal system of equations of that model, describing micropolar Cosserat-type elasticity, coincides in the principle with (34),(35). Notice that the Maxwell-Rayleigh model [16] leads also to the double dispersion. However, the dispersive terms (U_{TTTT} and U_{XXTT}) are different from those in (42), which means a different operator for the wave hierarchy. While (42) includes the space of derivatives of the wave operator in brackets, this model [16] indicates the role of time derivatives of the wave operator.

We focus now on possible nonlinear effects on the microstructural level motivated by experiments. In order to understand again the basic effects, the description of the macrolevel is kept linear, and only one term in the free energy function is added compared to (13) with $\beta = 0$. Suppose

$$W = \frac{1}{2}\alpha u_x^2 - A\psi u_x + \frac{1}{2}B\psi^2 + \frac{1}{2}C\psi_x^2 + \frac{1}{6}M\psi^3. \tag{43}$$

Then instead of (34), (35) we have

$$\rho\, u_{tt} = \alpha\, u_{xx} - A\, \psi_x, \tag{44}$$

$$I\, \psi_{tt} = C\, \psi_{xx} + M\, \psi_x \psi_{xx} - A\, u_x + B\, \psi. \tag{45}$$

Following the slaving procedure as before, we obtain

$$\psi_1 = \epsilon\, \alpha\, A\, I^* B^{-2} U_{XTT} - \epsilon\, \alpha\, A\, C^* B^{-2} U_{XXX} - \epsilon\, \delta^{\frac{1}{2}} A^2 M^* B^{-3} U_{XX} U_{XXX}, \tag{46}$$

where $M = l^2 M^*$. Due to the basic assumption about the microstructure, the nonlinearity is now described by high-order derivatives. The final governing equation is then

$$U_{TT} = \Big(1 - \frac{A^2}{\alpha B}\Big) U_{XX} + \delta \frac{A^2 I^*}{B^2}\Big(U_{TT} - \frac{C^*}{\alpha I^*} U_{XX}\Big)_{XX} - \delta^{\frac{3}{2}} \epsilon \frac{A^3 M^*}{\alpha B^3}\Big[\frac{1}{2}(U_{XX})^2\Big]_{XX}. \tag{47}$$

The effect of the nonlinearity is emphasized by the last term of equation (47). Its effect on wave profile distortion should be examined in detail. Obviously, the evolution equation could be derived on the basis of equation (47).

6 Discussion

The model equations demonstrate clearly the fundamental influence of a microstructure on the wave motion - changes in the wave speed, dispersive character of motion, and influence of a scale parameter. This permits us also to relate the experimental measurements directly to theoretically introduced parameters (cf. for example, the changes in wave speed calculated by Berezovski et al. [1].

We have found that various features of the microstructure induce corresponding terms in the governing equation (14). For instance, inertia yields the mixed derivative terms v_{xxtt}, v_{xxttt} , while dispersion v_{xxxx} and higher order dissipative term v_{xxxxt} result from the presence of the gradient of the micro-field. Dispersion is required for the existence of the bell-shaped solitary waves in an elastic microstructured medium. The mixed dispersive term v_{xxtt} decides, in particular, on the existence of either exact compression or tensile solitary wave solutions, see (17), (18), while mixed derivative higher order dissipative term $\alpha_6\, v_{xxttt}$ provides nonzero q_1 in Eq.(23), thus giving two stationary finite velocities of the solitary wave. Dispersion terms also account for the amplification of the kink-shaped wave while higher order dissipative terms are responsible for a saturation which prevents unbounded growth of the bell-shaped solitary wave.

The solutions of Eq.(14) allow to describe in an *explicit form* the amplification of both types of waves, as well as the selection of the solitary wave when its parameters tend to the *finite* values prescribed by the coefficients of the governing equation. The relationships between these parameters define the thresholds that separate the parameters of the initial solitary waves which will amplify or attenuate. The wave amplitude and velocity depend upon macro- and micro-properties of the elastic medium through explicit analytical relationships.

The asymptotic solution describing bell-shaped solitary wave selection may explain the transfer of strain energy by the microstructure. Consider a train of two bell-shaped solitary waves with initial amplitudes and velocities yielding attenuation of the first wave and amplification of the second. When they begin to move inside the elastic medium the microstructure absorbs energy from the first solitary wave and transmits it to the second one. The solution gives us the range of the microstructure parameters for which the energy transmission is possible.

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