THE TWO-DIMENSIONAL ANALYSIS OF THICKNESS-SHEAR VIBRATIONS OF A CONVEX-CONVEX QUARTZ PLATES

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Abstract. Low frequency piezoelectric quartz crystals working on the first overtone of thickness-shear mode are popular sources of a precious and highly stable frequencies. The analysis of such crystals is more complicated compared to flat crystals. Usually the energy trapping method is used instead of full three-dimensional analysis. In this paper the new theory of shells [1] is employed. The crystal plate is assumed to be a symmetrical shell. Two-dimensional anisotropic material tensors are obtained. Mode shape of the fundamental harmonic and anharmonic overtones are numerically calculated.

1 Introduction.

Quartz crystal industry generally work with two basic cuts for manufacturing crystals, working on Bulk Acoustic Waves (BAW): rotated Y-cuts (AT-cut, BT-cut, etc.) and doubly rotated cuts (SC-cut, MSC-cut, IT-cut, FC-cut, etc.). Rotated Y-cuts are more simple for both analyzing and manufacturing compared to doubly rotated cuts. This is so due to high anisotropy of doubly rotated crystal plates which causes all elastic tensor components to be non-zero. This causes conventional three-dimensional analysis to consume too much CPU resources. The two-dimensional analysis usually operates with displacements $\mathbf{u}_1$ and $\mathbf{u}_3$. The aim of the paper is to obtain the equation for thickness-shear vibrations of the symmetrical convex-convex lens, operating with turns ϕ_1 and ϕ_3. Because the rotated Y-axe, which is called mechanical, coincide with thickness direction, we will use following convention. Let the subscripts α, β, γ, δ take the values 1 and 3, but skip 2. Let the latins subscripts take values 1,2,3. Also denote thickness co-ordinate $y \equiv x_2$. Denote $\mathbf{n} = \mathbf{e}_2$ as the vector, normal to the middle surface. Three-dimensional variables will be used with carets. Further the following parameters will be used for the convex resonator: in undeformed state the material occupies volume Ω: $x_1 \in [-L_1, L_1]$, $x_3 \in [-L_3, L_3]$, $y \in [-h/2, h/2]$. The form of $h(x_1, x_3)$ will be specified later. The free boundary conditions are applied on the major surfaces and edges of the plate.

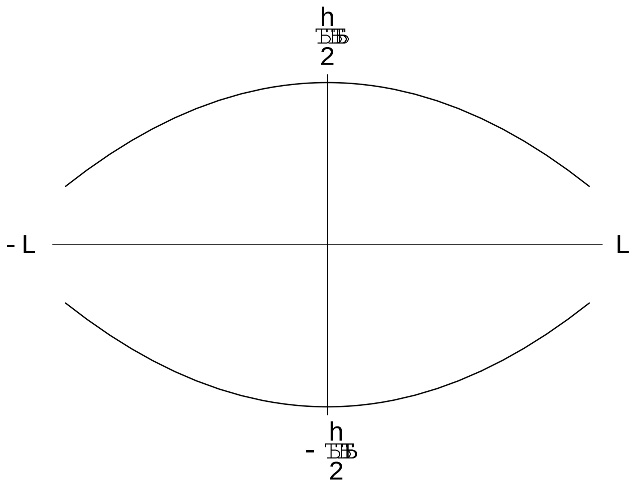

Figure 1: The crystal convex-convex resonator

2 The conventional form of linear piezoelectricity equations.

The equations of motion at the absence of external forces:

$$\hat{\nabla} \cdot \boldsymbol{\tau} = \rho \ddot{\mathbf{u}}, \quad \boldsymbol{\tau} = \boldsymbol{\tau}^T, \tag{1}$$

$$\hat{\nabla} \cdot \hat{\mathbf{D}} = 0, \quad \hat{\nabla} \cdot \mathbf{E}' = 0, \tag{2}$$

where $\hat{\nabla}$ is Hamilton operator (7), $\boldsymbol{\tau}$ is the stress tensor, ρ is the mass density, $\hat{\mathbf{u}}$ is the displacement vector in 3D-space, $\hat{\mathbf{D}}$ is the electrical induction, $\mathbf{E}'$ is the electrical field in the vacuum.

The piezoelectric constitutive equations

$$\boldsymbol{\tau} = \mathbf{C} \cdot \cdot \hat{\varepsilon} - \mathbf{E} \cdot \boldsymbol{\mathcal{P}}, \tag{3}$$

$$\hat{\mathbf{D}} = \boldsymbol{\mathcal{P}} \cdot \cdot \hat{\varepsilon} + \boldsymbol{\epsilon} \cdot \mathbf{E}, \tag{4}$$

where $\mathbf{E}$ is the electrical field in the crystal,

$$\mathbf{E} = -\hat{\nabla} \varphi, \tag{5}$$

φ is electric potential, $\hat{\varepsilon}$ is the tensor of the linear deformation

$$\hat{\varepsilon} = \frac{1}{2}(\hat{\nabla}\hat{\mathbf{u}} + \hat{\nabla}\hat{\mathbf{u}}^T), \tag{6}$$

$\mathbf{C}$ is the elasticity tensor, $\boldsymbol{\mathcal{P}}$ is the tensor of piezoelectric modulus, $\boldsymbol{\epsilon}$ is the dielectric tensor.

3 The equations of two-dimensional theory.

Let us consider the elastic body, occupied volume Ω. Let $x = (x_1, x_3, y) \equiv (x_\alpha, y)$ be material (Lagrangian) coordinates. Let $\mathbf{R}(x,t)$ be radius-vectors of the points in the reference configuration. The vector basis and reciprocal vector basis are defined by following equations:

$$\mathbf{R}_s = \frac{\partial \mathbf{R}(x,0)}{\partial x^s}, \qquad \mathbf{R}^s \cdot \mathbf{R}_m = \delta^s_m.$$

Since the linear theory is used, only one Hamilton operator for reference configuration is needed [7]:

$$\hat{\nabla} = \mathbf{R}^s \frac{\partial}{\partial x^s} \tag{7}$$

Let $\mathbf{r}(x_\alpha, t)$ be middle surface of the plate. The following interpolation will be used:

$$\mathbf{R}(x,t) = \mathbf{r}_(x_\alpha, t) + y\mathbf{n}.$$

Introduce the vector basis for this two-dimensional case:

$$\mathbf{r}_\alpha = \frac{\partial \mathbf{r}(x_\alpha, 0)}{\partial x^\alpha}, \qquad \mathbf{r}^\alpha \cdot \mathbf{r}_\beta = \delta^\alpha_\beta,$$

and Hamilton operator is defined by equation:

$$\nabla = \mathbf{r}^\alpha \frac{\partial}{\partial x^\alpha}.$$

Further the theory of shells [1], introduced by P.A. Zhilin, will be employed. The first metrical tensor (ortho-projector) $\mathbf{a}$ and the second metrical tensor $\mathbf{b}$ are defined by the equation

$$\mathbf{a} = \nabla \boldsymbol{r} = \mathbf{I} - \mathbf{n}\mathbf{n}, \qquad \mathbf{b} = -\nabla \mathbf{n} = \mathbf{0}, \tag{8}$$

where $\mathbf{I}$ is identity tensor. The last equation follows from the obvious fact, that if the shell is symmetrical, then the middle surface is flat. So, $\mathbf{n}$ does not depend on co-ordinates. Discriminant tensor $\mathbf{c}$ can be set by equation

$$\mathbf{c} = -\mathbf{a} \times \mathbf{n}, \tag{9}$$

and $\hat{\nabla}$ operator can be represented as

$$\hat{\nabla} = \nabla + \mathbf{n}\frac{\partial}{\partial y}. \tag{10}$$

Let the vector $\hat{\mathbf{u}} = \mathbf{R}(x,t) - \mathbf{R}(x,0)$ be three-dimensional displacement, the vector $\mathbf{u} = \mathbf{r}(x_\alpha,t) - \mathbf{r}(x_\alpha,0)$ be two-dimensional displacement of points in the middle plane. The simplest approximation for $\hat{\mathbf{u}}$ is

$$\hat{\mathbf{u}} = \mathbf{u} + y\mathbf{n} \times \boldsymbol{\phi}, \tag{11}$$

where $\boldsymbol{\phi}(x_\alpha,t)$ is the two-dimensional turn-vector. The tensor of elasticity $\mathbf{C}$ satisfies the condition of symmetry and apolarity

$$\mathbf{C}\cdot\cdot\mathbf{d} = \mathbf{d}\cdot\cdot\mathbf{C} \quad \forall\mathbf{d}, \qquad \mathbf{C}\cdot\cdot\mathbf{d}_1 = 0 \quad \forall\mathbf{d}_1 : \mathbf{d}_1 = -\mathbf{d}_1^T. \tag{12}$$

Decompose the stress tensor $\boldsymbol{\tau} = \boldsymbol{\tau}_m + \boldsymbol{\tau}_e$, where $\boldsymbol{\tau}_m$ is the mechanical part of the stress tensor and $\boldsymbol{\tau}_e$ is electrically induced one, by the following way:

$$\boldsymbol{\tau}_m = \mathbf{C}\cdot\cdot\hat{\boldsymbol{\varepsilon}} = \mathbf{C}\cdot\cdot\hat{\nabla}\hat{\mathbf{u}}^s = \mathbf{C}\cdot\cdot\hat{\nabla}\hat{\mathbf{u}} = \mathbf{C}\cdot\cdot(\nabla\mathbf{u} + \hat{\nabla}(y\mathbf{n}\times\boldsymbol{\phi})) =$$
$$\mathbf{C}\cdot\cdot(\nabla\mathbf{u} + \mathbf{n}(\mathbf{n}\times\boldsymbol{\phi}) + y\hat{\nabla}(\mathbf{n}\times\boldsymbol{\phi}) = \mathbf{C}\cdot\cdot(\boldsymbol{\varepsilon} + \mathbf{n}\boldsymbol{\gamma} + y\boldsymbol{\Phi}\cdot\mathbf{c}), \tag{13}$$

In (13) we have used the following notations:

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) \tag{14}$$

is the tensor of deformation in plane;

$$\boldsymbol{\gamma} = \mathbf{n}\times\boldsymbol{\phi} \tag{15}$$

is the vector of transversal shear;

$$\boldsymbol{\Phi} = \nabla\boldsymbol{\phi} \tag{16}$$

is the bending-twisting tensor. Define the electric field to be parallel to $\mathbf{n}$:

$$\mathbf{E}(x_\alpha,t) = -\frac{V(x_\alpha,t)}{h(x_\alpha)}\mathbf{n}, \qquad V = \int_{-h/2}^{h/2} \boldsymbol{\nabla}\varphi\cdot\mathbf{n}dy, \tag{17}$$

where V is the difference of electric potential between upper and lower points of the plate. On the electroded region of the plate $V = V_0 e^{-i\omega t}$ is applied voltage. In the case of shortened electrodes $V = 0$. The electrically induced part of the stress tensor

$$\boldsymbol{\tau}_e = -\mathbf{E}\cdot\boldsymbol{\mathcal{P}} = \frac{V}{h}\mathbf{n}\cdot\boldsymbol{\mathcal{P}}. \tag{18}$$

If the material is homogenous, then two-dimensional density at initial state is defined as

$$\rho_0 =< \hat{\rho}_0 >\equiv \int_{-h/2}^{h/2} \hat{\rho}_0 dy = \hat{\rho}_0 h(x_1,x_2). \tag{19}$$

From the symmetry of the plate with respect to the middle surface it follows, that the first inertia tensor and the second inertia tensor are, respectively

$$\rho_0\boldsymbol{\Theta}_1 = - < \hat{\rho}_0 z > \mathbf{c} = 0 \qquad \rho_0\boldsymbol{\Theta}_2 =< \hat{\rho}_0 z^2\mathbf{a} >= \hat{\rho}_0 h^3/12\mathbf{a}$$

Neglecting irreversible polarization, the following definition for the electric displacement vector can be given:

$$\hat{\mathbf{D}} = \mathbf{E} + 4\pi\mathbf{P}, \qquad \mathbf{P} = \frac{\partial(q\delta\mathbf{d})}{\partial\Omega}, \tag{20}$$

where $\mathbf{P}$ is polarization vector, q is total positive charge and $\delta\mathbf{d}$ is difference between centers of positive and negative charges within volume $\partial\Omega = \partial x_1 \partial x_3 \partial y$. Integrating the polarization vector along the thickness of the plate, obtain:

$$\int_{-h/2}^{h/2} \mathbf{P}\cdot\mathbf{n}dy = q\int_{-h/2}^{h/2} \frac{\partial^3 \delta\mathbf{d}}{\partial x_1 \partial x_3 \partial y}\cdot\mathbf{n}dy = \mathbf{n}\cdot\frac{\partial^2}{\partial x_1\partial x_3} q\int_{-h/2}^{h/2} d\delta\mathbf{d} = \frac{\partial^2(q(d^+ - d^-))}{\partial x_1 \partial x_3} = \frac{\partial^2(q^+d^+ + q^-d^-))}{\partial x_1\partial x_3} = \frac{\partial^2}{\partial x_1 \partial x_3}(q^{S+}\frac{h}{2} - q^{S-}\frac{h}{2})) = \frac{\partial^2(q^S h)}{\partial x_1\partial x_3},$$

where q^S is a surface charge value, d^+ and d^- are centers of positive and negative charges inside the cylinder $h\partial S$. At this point assume that h changes relatively slowly in planar co-ordinates. Now a new variable is introduced Q^S - density of surface charge:

$$Q^S = \frac{1}{h}\int_{-h/2}^{h/2} \mathbf{P}\cdot\mathbf{n}dy. \tag{21}$$

Let us define two-dimensional value of $\hat{\mathbf{D}}$:

$$D \equiv -\frac{1}{h}\int_{-h/2}^{h/2} \hat{\mathbf{D}}\cdot\mathbf{n}dy = \frac{V}{h} - 4\pi Q^S. \tag{22}$$

According to [1], the equations of motion are:

$$\nabla\cdot\mathbf{T} + \rho\mathbf{F} = \rho_0\ddot{\mathbf{u}}, \tag{23}$$

$$\nabla\cdot\mathbf{M} + \mathbf{T}_\times + \rho\mathbf{L} = \rho_0\frac{h^2}{12}\ddot{\boldsymbol{\phi}}, \tag{24}$$

where $\mathbf{T}$ and $\mathbf{M}$ are two-dimensional stress tensor and coupled stress tensor. The energy balance equation for piezoelectrical plate in linear case may be written as:

$$\rho\dot{F} = (\mathbf{T}\cdot\mathbf{a})\cdot\cdot\dot{\varepsilon} + (\mathbf{T}\cdot\mathbf{n})\cdot\dot{\gamma} + \mathbf{M}^T\cdot\cdot\dot{\mathbf{\Phi}} - D\dot{V}, \tag{25}$$

where F is the mass density of free energy. Let the initial stress is zero. Since we are interested in a linear theory only, we construct a homogenous quadratic form of F:

$$\rho F = \frac{1}{2}\boldsymbol{\varepsilon}\cdot\cdot\mathbf{C_1}\cdot\cdot\boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\gamma}\cdot\mathbf{\Gamma}\cdot\boldsymbol{\gamma} + \frac{1}{2}\mathbf{\Phi}\cdot\cdot\mathbf{C_3}\cdot\cdot\mathbf{\Phi} + \frac{1}{2}\epsilon V^2 + \boldsymbol{\varepsilon}\cdot\cdot\mathbf{C_2}\cdot\cdot\mathbf{\Phi} + \boldsymbol{\gamma}\cdot\mathbf{\Gamma_1}\cdot\cdot\boldsymbol{\varepsilon} + \boldsymbol{\gamma}\cdot\mathbf{\Gamma_2}\cdot\cdot\mathbf{\Phi} + V\boldsymbol{\mathcal{P}}_1\cdot\cdot\boldsymbol{\varepsilon} + V\boldsymbol{\mathcal{P}}_2\cdot\boldsymbol{\gamma} + V\boldsymbol{\mathcal{P}}_3\cdot\cdot\mathbf{\Phi} \tag{26}$$

From (25) and (26), Cauchy-Green relations follow:

$$\mathbf{T}\cdot\mathbf{a} = \frac{\partial\rho F}{\partial\boldsymbol{\varepsilon}} = \mathbf{C_1}\cdot\cdot\boldsymbol{\varepsilon} + \mathbf{C_2}\cdot\cdot\mathbf{\Phi} + \boldsymbol{\gamma}\cdot\mathbf{\Gamma_1} + V\boldsymbol{\mathcal{P}}_1, \tag{27}$$

$$\mathbf{N} \equiv \mathbf{T}\cdot\mathbf{n} = \frac{\partial\rho F}{\partial\boldsymbol{\gamma}} = \mathbf{\Gamma}\cdot\boldsymbol{\gamma} + \mathbf{\Gamma_1}\cdot\cdot\boldsymbol{\varepsilon} + \mathbf{\Gamma_2}\cdot\cdot\mathbf{\Phi} + V\boldsymbol{\mathcal{P}}_2, \tag{28}$$

$$\mathbf{M}^T = \frac{\partial\rho F}{\partial\mathbf{\Phi}} = \boldsymbol{\varepsilon}\cdot\cdot\mathbf{C_2} + \mathbf{C_3}\cdot\cdot\mathbf{\Phi} + \boldsymbol{\gamma}\cdot\mathbf{\Gamma_2} + V\boldsymbol{\mathcal{P}}_3, \tag{29}$$

$$D = -\frac{\partial\rho F}{\partial V} = -\epsilon V - \boldsymbol{\mathcal{P}}_1\cdot\cdot\boldsymbol{\varepsilon} - \boldsymbol{\mathcal{P}}_2\cdot\boldsymbol{\gamma} - \boldsymbol{\mathcal{P}}_3\cdot\cdot\mathbf{\Phi}. \tag{30}$$

In (27), (28) and (29) $\mathbf{C_1}$, $\mathbf{C_2}$, $\mathbf{C_3}$, $\mathbf{\Gamma}$, $\mathbf{\Gamma_1}$, $\mathbf{\Gamma_2}$, $\boldsymbol{\mathcal{P}}_1$, $\boldsymbol{\mathcal{P}}_2$, $\boldsymbol{\mathcal{P}}_3$ and ϵ are material tensors of the shell.

Equations (23), (24), (2) together with (27)-(30) completely specify the system of 6 equations, that can be solved. But for this it is necessary to derive two-dimensional material tensors, mentioned above.

4 Material tensors.

4.1 Representation of material tensors.

Let us introduce two orthogonal vector basises: the crystalline basis ($\mathbf{e}_1^0$, $\mathbf{e}_2^0$, $\mathbf{e}_3^0$) and the plate basis ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$). These basises are connected by the equations

$$\mathbf{e}_i = \mathbf{Q} \cdot \mathbf{e}_i^0, \qquad \mathbf{e}_i^0 = \mathbf{Q}^T \cdot \mathbf{e}_i. \tag{31}$$

$\mathbf{Q}$ is orthogonal transformation tensor. For doubly rotated quartz cuts

$$\mathbf{Q} = \mathbf{P}(\alpha_2 \mathbf{e}_1) \cdot \mathbf{P}(\alpha_1 \mathbf{e}_3), \tag{32}$$

where the notation

$$\mathbf{P}(\alpha \mathbf{m}) = (1 - cos\alpha)\mathbf{mm} + cos\alpha \mathbf{I} + sin\alpha \mathbf{m} \times \mathbf{I}$$

is adopted. For example, for SC-cut $\alpha_1 = 21°56'$, $\alpha_2 \approx 34°$. Usually the material tensor is defined in the crystalline basis

$$\mathbf{C} = C_{ijkl}^0 \mathbf{e}_i^0 \mathbf{e}_j^0 \mathbf{e}_k^0 \mathbf{e}_l^0, \tag{33}$$

where C_{ijkl}^0 are known from the experimental data. It is easy to represent this tensor in the plate basis:

$$\mathbf{C} = C_{ijkl}^0 (\mathbf{Q}^T \cdot \mathbf{e}_i)(\mathbf{Q}^T \cdot \mathbf{e}_j)(\mathbf{Q}^T \cdot \mathbf{e}_k)(\mathbf{Q}^T \cdot \mathbf{e}_l) = C_{ijkl} \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k \mathbf{e}_l. \tag{34}$$

The plate is vibrating in the vicinity of desired thickness-shear mode, so called, mode C. This means that ϕ_3 is much larger than other variables. This condition is automatically valid when singly-rotated crystals are considered. But, when the crystal is doubly rotated, it is useful to employ the procedure of co-ordinate system transformation, described in [5]. This transformation allows to neglect all other variables, but ϕ_3.

4.2 Material tensors for 2D shells.

Two-dimensional stress tensor, couples stress tensor and bending stress vector are expressed from three-dimensional stress tensor by equations

$$\mathbf{T} \cdot \mathbf{a} = < \mathbf{a} \cdot \boldsymbol{\tau} \cdot \mathbf{a} >, \quad \mathbf{M} = < \mathbf{a} \cdot \boldsymbol{\tau} \cdot \mathbf{c} z >, \quad \mathbf{N} = < \mathbf{a} \cdot \boldsymbol{\tau} \cdot \mathbf{n} > . \tag{35}$$

By using (22), (35), (13) and (18), and performing integration over the plate thickness, it is possible to obtain the relation between two-dimensional stress tensors and two-dimensional deformations (14), (15), (17) and V:

$$\mathbf{T} \cdot \mathbf{a} = h \mathbf{a} \cdot (\mathbf{C} \cdot \cdot \boldsymbol{\varepsilon}) \cdot \mathbf{a} + h \mathbf{a} \cdot (\mathbf{C} \cdot \cdot (\mathbf{n} \boldsymbol{\gamma})) \cdot \mathbf{a} + V \mathbf{a} \cdot (\mathbf{n} \cdot \boldsymbol{\mathcal{P}} \cdot \mathbf{a}) \tag{36}$$

$$\mathbf{N} = h \mathbf{a} \cdot (\mathbf{C} \cdot \cdot \boldsymbol{\varepsilon}) \cdot \mathbf{n} + h \mathbf{a} \cdot (\mathbf{C} \cdot \cdot (\mathbf{n} \boldsymbol{\gamma})) \cdot \mathbf{n} + V \mathbf{a} \cdot (\mathbf{n} \cdot \boldsymbol{\mathcal{P}} \cdot \mathbf{n}), \tag{37}$$

$$\mathbf{M} = \frac{h^3}{12} \mathbf{a} \cdot (\boldsymbol{C} \cdot \cdot (\boldsymbol{\Phi} \cdot \mathbf{c})) \cdot \mathbf{c}. \tag{38}$$

$$D = \frac{V}{h} \mathbf{n} \cdot \boldsymbol{\epsilon} \cdot \mathbf{n} - \mathbf{n} \cdot \boldsymbol{\mathcal{P}} \cdot \cdot \boldsymbol{\varepsilon} - \mathbf{n} \cdot \boldsymbol{\mathcal{P}} \cdot \cdot (\mathbf{n} \boldsymbol{\gamma}) \tag{39}$$

Comparing equations (27), (28), (29) and (30) with (36), (37), (38) and (39) it is possible to calculate the material tensors for the given h-profile shell:

$$\boldsymbol{\Gamma} = h \mathbf{C}_{\alpha \mathbf{2} \beta \mathbf{2}} \mathbf{e}_\alpha \mathbf{e}_\beta, \qquad \boldsymbol{\Gamma}_{\mathbf{1}} = h \mathbf{C}_{\alpha \beta \gamma \mathbf{2}} \mathbf{e}_\gamma \mathbf{e}_\alpha \mathbf{e}_\beta, \qquad \boldsymbol{\Gamma}_{\mathbf{2}} = 0, \tag{40}$$

$$\mathbf{C_1} = h\mathbf{a}\cdot\mathbf{C}, \qquad \mathbf{C_2} = 0, \qquad \mathbf{C_3} = \frac{h^3}{12}\mathbf{C_{i_1j_1lk_1}} a_{ji_1} c_{kk_1} c_{j_1i} \mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l, \tag{41}$$

$$\boldsymbol{\mathcal{P}}_1 = \mathbf{a}\cdot(\mathbf{n}\cdot\boldsymbol{\mathcal{P}}\cdot\mathbf{a}), \qquad \boldsymbol{\mathcal{P}}_2 = \mathbf{a}\cdot(\mathbf{n}\cdot\boldsymbol{\mathcal{P}}\cdot\mathbf{n}), \qquad \boldsymbol{\mathcal{P}}_3 = 0, \tag{42}$$

$$\epsilon = -\frac{1}{h}\mathbf{n}\cdot\boldsymbol{\epsilon}\cdot\mathbf{n} \tag{43}$$

If the crystal is electrically fixed, i.e. $D = 0$, then it is possible to introduce modified 2D tensors:

$$\mathbf{C_1'} = \mathbf{C_1} - \frac{\boldsymbol{\mathcal{P}}_1\otimes\boldsymbol{\mathcal{P}}_1}{\epsilon}, \qquad \boldsymbol{\Gamma'} = \boldsymbol{\Gamma} - \frac{\boldsymbol{\mathcal{P}}_2\otimes\boldsymbol{\mathcal{P}}_2}{\epsilon}, \qquad \boldsymbol{\Gamma_1'} = \boldsymbol{\Gamma_1} - \frac{\boldsymbol{\mathcal{P}}_2\otimes\boldsymbol{\mathcal{P}}_1}{\epsilon}. \tag{44}$$

5 Solution.

In order to construct the equation describing free mode C vibration, substitute (14)-(17) and known material tensors into Cauchy-Green relations. Using equations of motion (23) and (24), it is possible to write the system of 5 equations with 5 variables. Now we don't take into account electrical effects. Neglecting coupling with the thickness displacement u_2 and $\partial h/\partial x_1$, $\partial h/\partial x_3$, the following two-dimensional system of equations, describing thickness-shear vibrations is obtained:

$$C_{55}\frac{\partial^2\phi_1}{\partial {x_1}^2} + C_{33}\frac{\partial^2\phi_1}{\partial {x_3}^2} + 2C_{35}\frac{\partial^2\phi_1}{\partial x_1\partial x_3} - C_{15}\frac{\partial^2\phi_3}{\partial {x_1}^2} - C_{35}\frac{\partial^2\phi_1}{\partial {x_3}^2} \\ - (C_{13}+C_{55})\frac{\partial^2\phi_3}{\partial x_1\partial x_3} + \frac{12C_{46}}{h^2}\phi_3 - \frac{12C_{44}}{h^2}\phi_3 - \rho\ddot{\phi}_1 = 0, \tag{45}$$

$$C_{11}\frac{\partial^2\phi_3}{\partial {x_1}^2} + C_{55}\frac{\partial^2\phi_3}{\partial {x_3}^2} + 2C_{15}\frac{\partial^2\phi_3}{\partial x_1\partial x_3} - C_{15}\frac{\partial^2\phi_1}{\partial {x_1}^2} - C_{35}\frac{\partial^2\phi_1}{\partial {x_3}^2} \\ - (C_{13}+C_{55})\frac{\partial^2\phi_1}{\partial x_1\partial x_3} + \frac{12C_{46}}{h^2}\phi_1 - \frac{12C_{66}}{h^2}\phi_3 - \rho\ddot{\phi}_3 = 0 \tag{46}$$

The constants C_{mn}, appearing in (45) and (46) denote standard form for symmetric elastic tensor (34). Further simplification is based on the case $\phi_3 \gg \phi_1$, i.e. when we consider vibrations of mode "C". Thus, the following approximated equation is obtained:

$$C_{11}\frac{\partial^2\phi_3}{\partial {x_1}^2} + C_{55}\frac{\partial^2\phi_3}{\partial {x_3}^2} + 2C_{15}\frac{\partial^2\phi_3}{\partial x_1\partial x_3} - \frac{12C_{66}}{h^2}\phi_3 - \rho\ddot{\phi}_3 = 0. \tag{47}$$

This equation does not contain any information about the interference with "B"-mode ($\phi_1 \gg \phi_3$) and may give errors when computing high frequency modes. In this case it is better to use equations (45) and (46).

Let us specify the form of h. Assume the radius of the sphere $R_{sph} \gg h_0$ and, thus, the following expression may be used:

$$h = h_0\left(1 - \frac{x_1^2 + x_3^2}{h_0 R_{sph}}\right) \tag{48}$$

The solution may be approximately found as a finite sum

$$\phi_3(x_1, x_3) = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} A_{ij} U_i(x_1/L_1) U_j(x_3/L_3), \tag{49}$$

where $U_n(x)$ is the set of functions, orthogonal in the range $x \in [-1; 1]$.

The equations (45) and (46), or (47), if more simple case is considered, together with (49) allow to form $N_1 \times N_2$ matrix for coefficients A_{ij}. As usual, the vanishing determinant of this matrix corresponds to the eigen-mode. For more accurate results, it is necessary to use the correction factor $\pi^2/12$ when determining the eigenfrequencies.

The crystal lens with radius of 5cm, plate diameter 8mm and maximal thickness 1mm, made of "SC" cut of quartz, was analyzed. The shapes of eigenmodes are presented below.

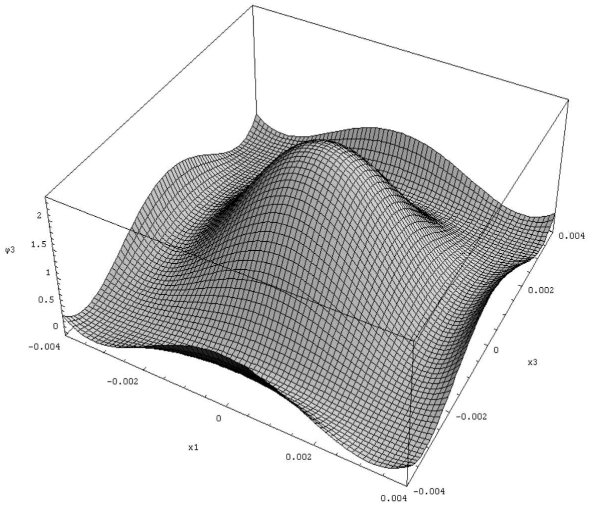

Figure 2: The main ("100") thickness-shear mode (1889Hz*m)

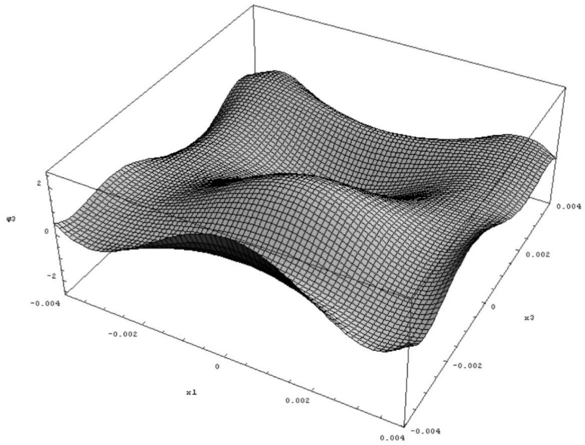

Figure 3: "101" anharmonic mode (2015Hz*m)

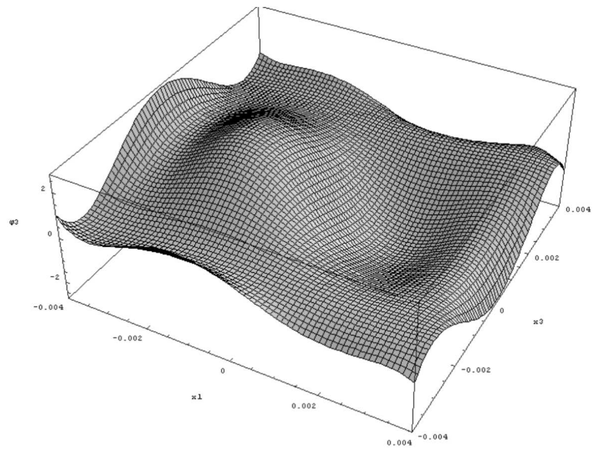

Figure 4: "110" anharmonic mode (2024Hz*m)

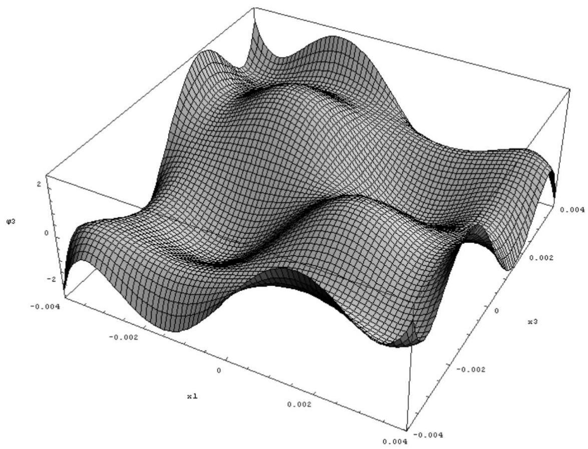

Figure 5: "111" anharmonic mode (2155Hz*m)

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