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Fundamental theorems and variational criterion in modified couple stress under multi-phase-lags and rotation

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ABSTRACT

Present study involves a new mathematical formulation of modified couple stress thermoelastic diffusion under multi-phase-lags model in rotating frame of reference. The governing equations for multi-phase-lags model in rotating frame of reference are obtained with the aid of modified Fourier's and Fick's laws and used to explore the energy, uniqueness, reciprocity theorems and variational criterion. Instantaneous concentrated body forces, heat and chemical potential sources along with moving heat and chemical potential sources are taken to illustrate the applications of the reciprocity theorem for a specific case. It has been observed that results obtained are dominated by the physical field. Some well-known results are also explored as special cases. The multi-phase-lags and rotation in modified couple stress material makes it more realistic as all the field quantities depend upon information from surrounding at a particular point. Physical views presented in this article may be useful for the design (composition, arrangement, device, etc.) new material, geophysics and other scientific domains. Also, the study has tremendous applications in material science, geomechanics, soil dynamics and electronic industry.

KEYWORDS

modified couple stress • thermoelastic diffusion • multi-phase-lags • rotating frame of reference • energy theorem uniqueness • reciprocity theorem • variation criterion

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Introduction

Over the last four decennary, the generalized theories of thermoelasticity has been established to remove the paradox of uncoupled and coupled theories. Lord and Shulman [1] (LS) proposed generalized thermoelastic model comprising one relaxation time whereas Green and Lindsay [2] (GL) rectified the LS model by involving another relaxation time to the respective equations and developed the new thermoelasticity theory. This modification allows that GL model to be dependent only on the strain-acceleration relation. To analyze the flow of heat without energy dissipation, Green and Naghdi [3] imported thermal displacement gradient to the humanistic Fourier law. Plethora of investigations based on the generalized theories of thermoelasticity have been considered in the work of Hetnarski and Ignaczak [4].

Phase lag with heat flow was first introduced by Tzou [5], who generalized it in the Cattaneo-Vernotte [6,7] (CV) formulation. The single-phase-lag heat transfer hypothesis is the term used most frequently to describe this idea. The dual-phase-lag (DPL) heat transfer theory was established by Tzou [8,9] be accumulating the temperature gradient

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and heat flux with two-phase-lag time constants. Roychoudhari [10] introduced a new model which being the extension of DPL theory and named as three-phase-lag thermoelastic model which involves Fourier's law of heat conduction being replaced by an approximation to an alteration to the Fourier's law with the consideration of three distinct phase lags for the heat flux vector, the temperature gradient and the thermal displacement gradient. Comprehensive work on thermal conductivity and phase lags has been given in book by Tzou [11].

In [12–15], Nowacki explored the thermoelastic diffusion model which is based on humanistic thermoelasticity theory. The thermoelastic diffusion theory with one relaxation time and two relaxation times have been proposed by Sherief et al. [16] and Kumar and Kansal [17], respectively.

A modified elasticity theory, such as couple stress theory, has been developed because classical theories of continuum mechanics are inadequate to describe the mechanical nature of micro/nano structures. Couple stress theory being an extended elasticity theory which in extension to the classical force per unit area, also involve the impact of couple per unit area on material volume. As shear stress no more requires to be conjugated in order to retain rational equilibrium, this permits for the probability of the asymmetric stress tensor. One of the non-classical theories as classical couple stress theory was developed by Mindlin and Tiersten [18], Toupin [19] and Koiter [20]. This theory involves extension to the two traditional Lame's constants for isotropic elastic material, the governing equation for isotropic materials also involves higher order material length scale characteristics. Modified couple stress (MCS) theory was formulated from couple stress theory by Yang et al. [21] depicted that couple stress tensor is symmetric, and strain energy is quadratic function of strain and symmetric curvatures MCS theory include single material length scale parameter contrary to two classical substantial constants. In micro/nanosystems, the theory has been broadly applied to examine the vibration, bending, and buckling of structures including beams, plates, and shells. Various investigators have predicted theoretically the size-dependent behavior of microstructures by using the MCS theory. Also, it has been documented by analyzing theoretical and experimental researches by Park and Geo [22]. Aouadi [23] presented reciprocity and uniqueness in anisotropic thermoelastic diffusion materials. Kumar, Kothari and Mukhopadhyay [24] proved a convolutional type variational criterion and reciprocity theorem based on the isotropic thermoelastic diffusion model. Kumar and Gupta [25,26] presented uniqueness reciprocity theorem and wave phenomena in thermoelastic diffusion with fractional derivative and also in different theories of thermoelasticity. Sarkar and Mukhopadhyay [27] presented energy and uniqueness theorem along with the variational criterion in thermoelastic model with three-phase-lags and memory dependent derivatives (MDD).

El-Karamany and Ezzat [28] established the uniqueness and reciprocal theorems for a linear anisotropic and inhomogeneous thermoelastic centro-symmetric solid in the frame of two-temperature Green–Naghdi theories. El-Karamany and Ezzat [29] investigated the uniqueness and reciprocal theorems without deploy of Laplace Transforms for the DPL thermoelasticity theory. Bala [30] explored basics theorems (uniqueness, continuous dependence and reciprocity) in relaxed micromorohic thermoelastic material. Al-Lehaibi [31] examined the variational criterion in two temperature thermoelastic material without energy dissipation. In [32], Paul and Mukhopadhyay presented a novel mathematical model on generalized thermoelastic diffusion theory and presented uniqueness, reciprocity and variational criterion in thermoelastic diffusion model.

Recently, some efforts have been done to modify the humanistic Fourier's law of heat conduction by considering time derivative of higher order by Abouelregal [33–35]. The refined multi-phase-lag (RPL) model finds many advantages in various problems. Zenkour [36–38] presented a distinct form of the RPL model to concern with thermoelastic replies of many various composite materials. Ezzat and Fayik [39] constructed thermoelastic diffusion with fractional order derivatives and proved uniqueness, reciprocity theorems and variational criterion. Kumar and Sharma [40] analysed presented some theorems in piezo-thermoelastic with mass diffusion. Sharma et al. [41] investigated basic theorems and plane waves in the context of thermoelastic diffusion using a multi-phase-lag model with temperature dependence.

Due to the fact that various large body like earth, moon and other planet have an angular velocity as well as earth itself act like a huge magnet, it is convincing to explore the fundamental basic theorems in modified couple stress thermoelastic diffusion medium rotating with angular velocity. Roy Choudhuri [42] examined the plane harmonic waves in an unbounded isotropic electromagnetic thermoelastic medium when the entire medium rotates with a uniform angular velocity. Kumar and Devi [43] presented the effect of hall current and rotation in modified couple stress generalized thermoelastic half-space due to ramp-type heating.

Ezzat [44] investigated fundamental solution in generalized magneto-thermoelasticity with two relaxation times for perfect conductor cylindrical region. Ezzat and Youssef [45] established Stokes' first problem for an electro-conducting micropolar fluid with thermoelectric properties. Sharma et al. [46] established the plane waves and fundamental solution in an electro microstretch of elastic solids. Ezzat et al. [47] examined the equations of generalized thermo-viscoelasticity for an isotropic medium with variable thermal conductivity and fractional-order heat transfer. Sharma and Khator [48,49] examined some problems of power generation due to renewable sources and also explored micro-grid planning in the renewable inclusive prosumer market. Panja et al. [50] investigated a thermoelastic model with higher order time derivatives for a crack in a rotating solid. Bibi et al. [51] established the propagation and refection of thermoelastic wave in a rotating nonlocal fractional order porous medium under Hall current influence. El-Sapa et al. [52] investigated the influence of rotation and viscosity on generalized conformable fractional micropolar thermoelasticity with two temperatures.

In this work, MCS thermoelastic diffusion with multi-phase-lags model along with rotating frame of reference has been presented. This model has been applied to explore fundamental energy theorems, uniqueness and reciprocity and variational criterion. These theorems are impacted by the various physical field variables forming the model. Applications of reciprocity theorem for body forces, heat sources and chemical potential sources are taken for specific case. Some unique cases are deduced and compared with the well-established results.

Governing equations

Consider the domain V of three dimensional space ($\mathbb{R}^3$) which is bounded by the piecewise smooth surface A in modified couple stress thermoelastic diffusion under multi-phase-lag model in rotating frame of reference. Let $\boldsymbol{x} = (x_1, x_2, x_3)$ be position of state variables and time variable $t \in \bar{V} \times [0, \infty)$, where, $\bar{V} = V \cup A$.

Then the preliminary equations by following Kumar and Kansal [17], Zenkour [36] given as:

1. stress-strain-temperature-chemical potential relation:

$$t_{ij} = 2\mu e_{ij} - \alpha e_{kij} \chi_{lk,l} + \delta_{ij}(\lambda_0 e_{kk} - \gamma_1 T - \gamma_2 P); \tag{1}$$

2. equation of motion:

$$t_{ij,j} + \rho F_i = \rho \ddot{u}_i + \rho(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u}))_i + \rho(2\boldsymbol{\Omega} \times \dot{\boldsymbol{u}})_i; \tag{2}$$

3. displacement-strain relation:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}); \tag{3}$$

4. energy equation:

$$-q_{i,i} + Q = \rho T_0 \dot{S}; \tag{4}$$

5. modified Fourier law:

$$\left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_q^r}{r!} \frac{\partial^r}{\partial t^r}\right) q_i = -K\left(1 + \sum_{r=1}^{M^0} \frac{\tau_T^r}{r!} \frac{\partial^r}{\partial t^r}\right) T_{,i}; \tag{5}$$

6. entropy-strain-temperature-chemical potential relation:

$$\rho T_0 \dot{S} = l_1 T_0 \dot{T} + \gamma_1 T_0 \dot{e}_{kk} + d T_0 \dot{P}; \tag{6}$$

7. mass concentration law:

$$-\eta_{i,i} + A_0 = \dot{C}; \tag{7}$$

8. equation of chemical potential:

$$\left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_\eta^r}{r!} \frac{\partial^r}{\partial t^r}\right) \eta_i = -D\left(1 + \sum_{r=1}^{M^0} \frac{\tau_P^r}{r!} \frac{\partial^r}{\partial t^r}\right) P_{,i}; \tag{8}$$

9. mass concentration-chemical potential-strain-temperature relation:

$$C = nP + \gamma_2 e_{kk} + dT, \tag{9}$$

where

$$\lambda_0 = \lambda - \frac{\beta_2^2}{b}, \gamma_1 = \beta_1 + \frac{a}{b}\beta_2, \gamma_2 = \frac{\beta_2}{b}, d = \frac{a}{b}, n = \frac{1}{b}, l_1 = \frac{\rho C_E}{T_0} + \frac{a^2}{b}. \tag{10}$$

Here, the medium is rotating with angular velocity $\boldsymbol{\Omega} = \Omega\hat{\nu}$, where $\hat{\nu}$ is the unit vector along the axis of rotation and the equations of the motion (2) include two additional terms namely:

1. the centripetal acceleration $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u})$ due to time-varying motion;
2. the Coriolis acceleration $2\boldsymbol{\Omega} \times \dot{\boldsymbol{u}}$.

In the Eqs. (1)–(10), λ and μ are Lame's constants, ρ is the density, α is couple stress parameter, C_E is the specific heat at the constant strain, q_i are the components of heat flux vector $\boldsymbol{q}$, F_i, u_i being the components of body force per unit mass and the components of the displacement vector $\boldsymbol{u}$ respectively, χ_{ij} is symmetric curvature, K, S, P and D related the thermal conductivity, entropy per unit mass , chemical potential and thermal diffusivity coefficient respectively, $t_{ij}(= t_{ji})$, $e_{ij}(= e_{ji})$ ($i, j = 1, 2, 3$) being the component of the stress tensor and the component of the strain tensor respectively, e_{kij} is alternate tensor, $T(= \Theta\text{-}T_0)$, where T, Θ and T_0 concern to temperature increment, the absolute temperature of medium and the reference temperature of the body respectively chosen such that, $|T/T_0| << 1$, $\beta_1 = (3\lambda + 2\mu)\alpha_t$, $\beta_2 = (3\lambda + 2\mu)\alpha_c$, where α_t, α_c correspond to the coefficient of thermal linear expansion and diffusion expansion coefficient respectively; δ_{ij} is Kronecker's symbols, Q and A_0 are source of heat and source of chemical potential respectively, a is coefficient which describes measure of thermoelastic diffusion, b is coefficient of mass diffusion, thermal relaxation times with τ_q and $\tau_T \geq 0$ and diffusion relaxation times with τ_η and $\tau_P \geq 0$, δ^0 is constant taking the values 0 or 1, dot$(\dot{})$ denotes time differentiation; N^0 and M^0 are finite positive integers.

Energy theorem

Statement

Conservation of energy for every time and every region V of $(\mathbb{R}^3)$ is of the profile [27]:

$$\frac{d}{dt}\left(K^* + W + P^* + M^* + d\int_V TPdV\right) = \tau_1^*\left[\int_A TT_{,i}n_i dA - \int_V T_{,i}T_{,i}dV\right] + \tau_2^*\left[\int_A PP_{,i}n_i dA - \right.$$
$$\left. - \int_V P_{,i}P_{,i}dV\right] + \rho\int_V F_i u_i dV + \int_A h_i u_i dA + \frac{1}{T_0}\int_V QTdV + \int_V A_0 PdV - \tag{11}$$
$$- \rho\int_V (\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u}))_i \dot{u}_i\, dV - \rho\int_V (2\boldsymbol{\Omega} \times \dot{\boldsymbol{u}})_i\, \dot{u}_i dV,$$

where K^*, W, P^* and M^* indicates the kinetic energy, the isothermal strain energy, heat potential and mass (diffusion) potential respectively and:

$$\tau_1^* = \frac{K\tau_T^*}{T_0\tau_q^*},\ \tau_2^* = \frac{D\tau_P^*}{\tau_\eta^*},\ \tau_q^* = \left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_q^r}{r!}\frac{\partial^r}{\partial t^r}\right),\ \tau_T^* = \left(1 + \sum_{r=1}^{M^0} \frac{\tau_T^r}{r!}\frac{\partial^r}{\partial t^r}\right),$$
$$\tau_\eta^* = \left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_\eta^r}{r!}\frac{\partial^r}{\partial t^r}\right),\ \tau_P^* = \left(1 + \sum_{r=1}^{M^0} \frac{\tau_P^r}{r!}\frac{\partial^r}{\partial t^r}\right). \tag{12}$$

Proof

Let the Eqs. (1) and (12) required with the subsequent boundary restrictions:

$$t_{ij}n_j = h_i(\boldsymbol{x},t), T(\boldsymbol{x},t) = \upsilon(\boldsymbol{x},t), P(\boldsymbol{x},t) = \omega(\boldsymbol{x},t),\ \boldsymbol{x} \in A,\ \mathrm{t} > 0, \tag{13}$$

where $h_i(\boldsymbol{x},t)$ is component of surface traction, $\omega(\boldsymbol{x},t)$ is chemical potential and $\upsilon(\boldsymbol{x},t)$ is heating of the surface to the temperature, n_j is the outward unit normal of A.

Multiplying Eq. (2) by $\dot{u}_i$ and integrating over region V, yields:

$$\int_V (t_{ij,j} + \rho F_i)\dot{u}_i dV = \int_V \rho\ddot{u}_i\, \dot{u}_i dV + \int_V \rho\big(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u})\big)_i \dot{u}_i\, dV + \int_V \rho(2\boldsymbol{\Omega} \times \dot{\boldsymbol{u}})_i\, \dot{u}_i dV. \tag{14}$$

In light of the relation $t_{ij,j}\dot{u}_i = (t_{ij}\dot{u}_i)_{,j} - t_{ij}\dot{u}_{i,j}$ and Gauss's divergence theorem, Eq. (14) with the aid of Eq. (3) reduces to:

$$\int_V \rho F_i\dot{u}_i dV + \int_A h_i\dot{u}_i\, dA = \int_V \rho\ddot{u}_i\dot{u}_i\, dV + \int_V \rho\big(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u})\big)_i \dot{u}_i\, dV +$$
$$+ \int_V \rho(2\boldsymbol{\Omega} \times \dot{\boldsymbol{u}})_i\, \dot{u}_i dV + \int_V t_{ij}\, \dot{e}_{ij} dV. \tag{15}$$

Incorporating Eqs. (1) and (15), yields:

$$\rho\int_V F_i\dot{u}_i dV + \int_A h_i\dot{u}_i\, dA = \rho\int_V \ddot{u}_i\dot{u}_i\, dV + \rho\int_V \big(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u})\big)_i \dot{u}_i\, dV +$$
$$+\rho\int_V (2\boldsymbol{\Omega} \times \dot{\boldsymbol{u}})_i\, \dot{u}_i dV + \int_V [2\mu e_{ij} - \alpha e_{kij}\chi_{lk,l} + \delta_{ij}(\lambda_0 e_{kk} - \gamma_1 T - \gamma_2 P)]\, \dot{e}_{ij} dV. \tag{16}$$

Introducing the following functions is isothermal strain energy:

$$W = \mu\int_V e_{ij}e_{ij}dV + \frac{\lambda_0}{2}\int_V e_{kk}e_{kk}dV, \tag{17}$$

$$K^* = \frac{\rho}{2}\int_V \dot{u}_i\dot{u}_i\, dV. \tag{18}$$

Equation (16) with the aid of Eqs. (17) and (18) reduces:

$$\rho\int_V F_i\dot{u}_i dV + \int_A h_i\dot{u}_i\, dA = \frac{d}{dt}W + \frac{d}{dt}K^* + \rho\int_V \big(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u})\big)_i \dot{u}_i\, dV +$$
$$+\rho\int_V (2\boldsymbol{\Omega} \times \dot{\boldsymbol{u}})_i\, \dot{u}_i dV - \int_V \gamma_1 T\dot{e}_{kk} dV - \int_V \gamma_2 P\, \dot{e}_{kk} dV, \tag{19}$$

where $\int_V e_{kij}\chi_{lk,l}\,\dot{e}_{ij}dV = 0$, as $\dot{e}_{ij} = \dot{e}_{ji}$ and $e_{ijk} = -e_{ikj}$, $\frac{1}{2}\left(\int_V e_{kij}\chi_{lk,l}\,\dot{e}_{ij}dV +\right.$ $\left.+\int_V e_{kji}\chi_{lk,l}\,\dot{e}_{ji}dV\right) = 0$, $\int_V \ddot{u}_i\dot{u}_i\,dV = \frac{1}{2}\frac{d}{dt}\left(\int_V \dot{u}_i\dot{u}_i\,dV\right)$.

In Eq. (19), the first two integrals on left hand sides encloses the explanation of motion (the body force F_i, surface traction t_i) whereas the last two integrals of right-hand sides of Eq. (19) encloses the explanation such as heat source and mass diffusion source with heating of the body.

Equation (5) with the aid of Eqs. (4) and (6) can be written as:

$$\left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_q^r}{r!}\frac{\partial^r}{\partial t^r}\right)\left(l_1 T_0\dot{T} + \gamma_1 T_0\dot{e}_{kk} + dT_0\dot{P} - Q\right) = K\left(1 + \sum_{r=1}^{M^0} \frac{\tau_T^r}{r!}\frac{\partial^r}{\partial t^r}\right)T_{,ii}\,. \tag{20}$$

Multiplying Eq. (20) with T and integrating over the region V, we get:

$$\tau_q^*\left[\int_V l_1 T_0\dot{T}TdV + \int_V \gamma_1 T_0\dot{e}_{kk}TdV + \int_V dT_0\dot{P}TdV - \int_V QTdV\right] = K\,\tau_T^*\int_V T_{,ii}TdV, \tag{21}$$

where

$$\tau_q^* = \left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_q^r}{r!}\frac{\partial^r}{\partial t^r}\right),\ \tau_T^* = \left(1 + \sum_{r=1}^{M^0} \frac{\tau_T^r}{r!}\frac{\partial^r}{\partial t^r}\right). \tag{22}$$

In light of the relation $T\,T_{,ii} = \left(T\,T_{,i}\right)_{,i} - T_{,i}T_{,i}$ and dividing by T_0, Eq. (21) can be written as:

$$\tau_q^*\left[l_1\int_V \dot{T}TdV + \gamma_1\int_V \dot{e}_{kk}TdV + d\int_V \dot{P}TdV - \frac{1}{T_0}\int_V QTdV\right] = \frac{K}{T_0}\tau_T^*\int_V\left(\left(T\,T_{,i}\right)_{,i} - T_{,i}T_{,i}\right)dV. \tag{23}$$

Using divergence theorem in Eq. (23), yields:

$$\begin{aligned}\tau_q^*\left[l_1\int_V \dot{T}TdV + \gamma_1\int_V \dot{e}_{kk}TdV + d\int_V \dot{P}TdV - \frac{1}{T_0}\int_V QTdV\right] = \frac{K}{T_0}\tau_T^*\int_A T\,T_{,i}n_i dA -\\ -\frac{K}{T_0}\tau_T^*\int_V T_{,i}T_{,i}dV.\end{aligned} \tag{24}$$

Introducing the following function:

$$P^* = \frac{l_1}{2}\int_V T^2 dV. \tag{25}$$

Equation (24) with the help of Eq. (25) takes the form:

$$\tau_q^*\left[\frac{dP^*}{dt} + \gamma_1\int_V \dot{e}_{kk}TdV + d\int_V \dot{P}TdV - \frac{1}{T_0}\int_V QTdV\right] = \frac{K}{T_0}\tau_T^*\int_A T\,T_{,i}n_i dA - \frac{K}{T_0}\tau_T^*\int_V T_{,i}T_{,i}dV. \tag{26}$$

Incorporating Eqs. (7) and (8) in Eq. (9) determine:

$$\left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_\eta^r}{r!}\frac{\partial^r}{\partial t^r}\right)\left(n\dot{P} + \gamma_2\dot{e}_{kk} + d\dot{T} - A_0\right) = D\left(1 + \sum_{r=1}^{M^0} \frac{\tau_P^r}{r!}\frac{\partial^r}{\partial t^r}\right)P_{,ii}. \tag{27}$$

Multiplying Eq. (27) by P and integrating over the region V, we obtain:

$$\tau_\eta^*\left[n\int_V \dot{P}PdV + \gamma_2\int_V P\,\dot{e}_{kk}dV + d\int_V \dot{T}PdV - \int_V A_0 PdV\right] = D\tau_P^*\int_V P_{,ii}PdV, \tag{28}$$

where

$$\tau_\eta^* = \left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_\eta^r}{r!}\frac{\partial^r}{\partial t^r}\right);\ \tau_P^* = \left(1 + \sum_{r=1}^{M^0} \frac{\tau_P^r}{r!}\frac{\partial^r}{\partial t^r}\right). \tag{29}$$

In light of the relation $PP_{,ii} = \left(P\,P_{,i}\right)_{,i} - P_{,i}P_{,i}$ and applying divergence theorem in Eq. (28) can be written as:

$$\begin{aligned}\tau_\eta^*\left[n\int_V \dot{P}PdV + \gamma_2\int_V P\,\dot{e}_{kk}dV + d\int_V \dot{T}PdV - \int_V A_0 PdV\right] = D\tau_P^*\int_A P\,P_{,i}n_i dA -\\ -D\tau_P^*\int_V P_{,i}P_{,i}dV.\end{aligned} \tag{30}$$

Introducing the following function:

$$M^* = \frac{n}{2}\int_V P^2 dV. \tag{31}$$

With the aid of Eq. (31), Eq. (30) can be written as:

$$\tau_\eta^*\left[\frac{dM^*}{dt} + \gamma_2\int_V P\,\dot{e}_{kk}dV + d\int_V \dot{T}PdV - \int_V A_0 PdV\right] = D\tau_P^*\int_A P\,P_{,i}n_i dA - D\tau_P^*\int_V P_{,i}P_{,i}dV. \tag{32}$$

Equation (19) with the aid of Eqs. (26) and (32) can be written as Eq. (11).

Uniqueness theorem

Theorem

Consider the domain V of three-dimensional space ($\mathbb{R}^3$) which is bounded by the piecewise smooth surface A in linear modified couple stress isotropic thermoelastic diffusion under multi-phase-lag model in rotating frame of reference. There is only one solution of functions: $u_i\,(\boldsymbol{x},t)$, $T(\boldsymbol{x},t)$, $P(\boldsymbol{x},t)$ of class $\mathrm{C}^{(m)}\,(m \geq 2)$ and $t_{ij}\,(\boldsymbol{x},t)$,$e_{ij}\,(\boldsymbol{x},t)$ of class $\mathrm{C}^{(l)}$,for $\boldsymbol{x} \in V + A$ possessing coordinates $\boldsymbol{x} = (x_1, x_2, x_3)$ at $t \geq 0$ that satisfy Eqs. (1)–(9) contingent to the border restrictions:

$u_i\,(\boldsymbol{x},t)$=$U_i\,(\boldsymbol{x},t)$, $T(\boldsymbol{x},t)$=$\upsilon(\boldsymbol{x},t)$, $P(\boldsymbol{x},t)$=$\omega(\boldsymbol{x},t)$, $\boldsymbol{x} \in A$, $t > 0$, (33)

and the initial conditions at t= 0:

$u_i\,(\boldsymbol{x},0)$= $u_{0i}(\mathbf{x})$, $T(\boldsymbol{x},0)$=$T_0(\boldsymbol{x})$, $P(\boldsymbol{x},0)$=$P_0(\boldsymbol{x})$, for $\boldsymbol{x} \in V$, (34)

$\frac{\partial^k u_i(\boldsymbol{x},0)}{\partial t^k} = u_{0i}^{(k)}(\boldsymbol{x}), k = 1,2,3,\dots,\max\{N^0, M^0\}$, for $\boldsymbol{x} \in V$, (35)

$\frac{\partial^k T(\boldsymbol{x},0)}{\partial t^k} = T_0^{(k)}(\boldsymbol{x}), k = 1,2,3,\dots,\max\{N^0, M^0\}$, for $\boldsymbol{x} \in V$, (36)

$\frac{\partial^k P(\boldsymbol{x},0)}{\partial t^k} = P_0^{(k)}(\boldsymbol{x}), k = 1,2,3,\dots,\max\{N^0, M^0\}$, for $\boldsymbol{x} \in V$, (37)

where $u_{0i}(\boldsymbol{x})$, $u_{0i}^{(k)}(\boldsymbol{x}), T_0(\boldsymbol{x}), T_0^{(k)}(\boldsymbol{x}), P_0(\boldsymbol{x})$ and $P_0^{(k)}(\boldsymbol{x})$ are known functions.

Let the Laplace transformation of all field variables exists, and physical parameters fulfill the disparities: $\lambda_0 > 0$, $\mu > 0$, $\rho > 0$, $K > 0$, $C_E > 0, T_0 > 0, D > 0$, $d > 0$, $n > 0$, $\tau_T > 0, \tau_q > 0, \tau_P > 0, \tau_\eta > 0$.

Solution

Let $u_i^{(1)}$,$T^{(1)}$,$P^{(1)}$,... and $u_i^{(2)}$,$T^{(2)}$,$P^{(2)}$,...are two set of solution of Eqs. (1)–(9) with homogeneous border and initial restrictions. Let us take:

$\Pi^0 = \{u_i$=$u_i^{(1)} - u_i^{(2)}$, $T = T^{(1)} - T^{(2)}$, $P = P^{(1)} - P^{(2)}\}$. (38)

The set Π^0 satisfy the governing Eqs. (1)–(9) without body force, heat source and mass diffusion source and also functions u_i, T and P fulfill the homogeneous border and initial restrictions:

$u_i\,(\boldsymbol{x},t) = 0$, $T(\boldsymbol{x},t) = 0$, $P(\boldsymbol{x},t) = 0$, $\boldsymbol{x} \in A$, $t > 0$, (39)

$u_i\,(\boldsymbol{x},0) = 0$, $T(\boldsymbol{x},0) = 0$, $P(\boldsymbol{x},0) = 0$, for $\boldsymbol{x} \in V$, (40)

$\frac{\partial^k u_i(\boldsymbol{x},0)}{\partial t^k} = 0, k = 1,2,3,\dots,\max\{N^0, M^0\}$, for $\boldsymbol{x} \in V$, (41)

$\frac{\partial^k T(\boldsymbol{x},0)}{\partial t^k} = 0, k = 1,2,3,\dots,\max\,\{N^0, M^0\}$, for $\boldsymbol{x} \in V$, (42)

$\frac{\partial^k P(\boldsymbol{x},0)}{\partial t^k} = 0, k = 1,2,3,\dots,\max\,\{N^0, M^0\}$, for $\boldsymbol{x} \in V$. (43)

Let the Laplace transform be defined as:

$\overline{f}(\boldsymbol{x}, \mathrm{s}) = L(f(\boldsymbol{x},t))$=$\int_0^\infty f(\boldsymbol{x},t)e^{-st}dt$; s > 0. (44)

Applying the Laplace transform on Eqs. (1)–(9), (39) and (40) after removing the body forces, thermal source and chemical potential source and omitting the bar for further simplification yield:

$$t_{ij} = 2\mu e_{ij} - \alpha e_{kij}\chi_{lk,l} + \delta_{ij}(\lambda_0 e_{kk} - \gamma_1 T - \gamma_2 P), \tag{45}$$

$$t_{ij,j} = \rho s^2 u_i, \tag{46}$$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{47}$$

$$-q_{i,i} = \rho T_0 s S, \tag{48}$$

$$\left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_q^r}{r!} s^r\right) q_i = -K \left(1 + \sum_{r=1}^{M^0} \frac{\tau_T^r}{r!} s^r\right) T_{,i}, \tag{49}$$

$$\rho T_0 s S = l_1 T_0 s T + \gamma_1 T_0 s e_{kk} + d T_0 s P, \tag{50}$$

$$-\eta_{i,i} = s C, \tag{51}$$

$$\left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_\eta^r}{r!} s^r\right) \eta_i = -D \left(1 + \sum_{r=1}^{M^0} \frac{\tau_P^r}{r!} s^r\right) P_{,i}, \tag{52}$$

$$C = nP + \gamma_2 e_{kk} + dT, \tag{53}$$

$$u_i(\boldsymbol{x}, s) = 0, T(\boldsymbol{x}, s) = 0, P(\boldsymbol{x}, s) = 0, \boldsymbol{x} \in A, \tag{54}$$

$$u_i\,(\boldsymbol{x}, 0) = 0, T(\boldsymbol{x}, 0) = 0, P(\boldsymbol{x}, 0) = 0, \boldsymbol{x} \in V. \tag{55}$$

Consider the integral:

$$\int_V t_{ij} e_{ij} dV = \int_V t_{ij} u_{i,j} dV = \int_V \left(t_{ij} u_i\right)_{,j} dV - \int_V t_{ij,j} u_i dV. \tag{56}$$

With the help of divergence theorem and owing to Eq. (54), yield:

$$\int_V \left(t_{ij} u_i\right)_{,j} dV = \int_A u_i t_{ij} n_j dA = 0. \tag{57}$$

Equation (56) with the aid of Eq. (57) takes form:

$$\int_V t_{ij} e_{ij} dV + \int_V t_{ij,j} u_i dV = 0. \tag{58}$$

Equation (58) with the aid of Eqs. (45) and (46) can be expressed in the form:

$$\int_V [2\mu e_{ij} - \alpha e_{kij} \chi_{lk,l} + \delta_{ij}(\lambda_0 e_{kk} - \gamma_1 T - \gamma_2 P)] e_{ij} dV + \int_V \rho s^2 u_i^2 dV = 0. \tag{59}$$

To determine, $\gamma_1 \int_V T e_{kk} dV$, following Biot [53], let us introduce a vector function ϕ_i connected with entropy through relation:

$$q_i = T_0 \dot{\phi}_i\,, \ \rho S = -\phi_{i,i}. \tag{60}$$

Invoking Eq. (60) in Eqs. (49) and (50) respectively and assuming that $\phi_i(\boldsymbol{x}, 0) = \dot{\phi}_i(\boldsymbol{x}, 0) = 0$, we get respectively:

$$\left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_q^r}{r!} s^r\right) s T_0 \phi_i = -K \left(1 + \sum_{r=1}^{M^0} \frac{\tau_T^r}{r!} s^r\right) T_{,i}, \tag{61}$$

$$-T_0 s \phi_{i,i} = l_1 T_0 s T + \gamma_1 T_0 s e_{kk} + d T_0 s P. \tag{62}$$

Multiply both sides of Eq. (61) by ϕ_i and integrating over the region V of body, yields:

$$\int_V \left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_q^r}{r!} s^r\right) s T_0 \phi_i^2 dV = -K \int_V \left(1 + \sum_{r=1}^{M^0} \frac{\tau_T^r}{r!} s^r\right) T_{,i} \phi_i dV. \tag{63}$$

With the help of divergence theorem and owing to Eq. (54), Eq. (63) can be written as:

$$\int_V \left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_q^r}{r!} s^r\right) s T_0 \phi_i^2 dV = K \int_V \left(1 + \sum_{r=1}^{M^0} \frac{\tau_T^r}{r!} s^r\right) T \phi_{i,i} dV. \tag{64}$$

Incorporating Eq. (62) in Eq. (64), gives:

$$-\gamma_1 \int_V T e_{kk} dV = \frac{1}{K\left(1 + \sum_{r=1}^{M^0} \frac{\tau_T^r}{r!} s^r\right)} \int_V \left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_q^r}{r!} s^r\right) s T_0 \phi_i^2 dV + \int_V l_1 T^2 dV + \int_V P d T dV. \tag{65}$$

To determine $\gamma_2 \int_V P e_{kk} dV$, let us introduce the vector function N_i defined as:

$$\eta_i = \dot{N}_i\,, C = -N_{i,i}. \tag{66}$$

Substituting the value of η_i and C from Eq. (66) in Eqs. (52) and (53) respectively and assuming that $N_i(\boldsymbol{x}, 0) = \dot{N}_i(\boldsymbol{x}, 0) = 0$, we obtain respectively:

$$\left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_\eta^r}{r!} s^r\right) s N_{,i} = -D \left(1 + \sum_{r=1}^{M^0} \frac{\tau_P^r}{r!} s^r\right) P_{,i}, \tag{67}$$

$$-N_{i,i} = nP + \gamma_2 e_{kk} + dT. \tag{68}$$

Multiplying both sides of Eq. (67) by N_i and integrating over the region of the body, we obtain:

$$\int_V \left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_\eta^r}{r!} s^r\right) s N_i^2 dV = -D \int_V \left(1 + \sum_{r=1}^{M^0} \frac{\tau_P^r}{r!} s^r\right) P_{,i} N_i dV. \tag{69}$$

With the help of divergence theorem and owing to Eq. (54), Eq. (69) can be written as:

$$\int_V \left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_\eta^r}{r!} s^r\right) s N_i^2 dV = \mathrm{D} \int_V \left(1 + \sum_{r=1}^{M^0} \frac{\tau_P^r}{r!} s^r\right) P N_{i,i} dV. \tag{70}$$

Using Eq. (68) in Eq. (70), yields:

$$-\gamma_2 \int_V P e_{kk} dV = \frac{1}{D\left(1+\sum_{r=1}^{M^0} \frac{\tau_P^r}{r!} s^r\right)} \int_V \left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_\eta^r}{r!} s^r\right) s N_i^2 dV + \int_V nP^2 dV + \int_V PdTdV. \quad (71)$$

Using the value of $\gamma_1 \int_V Te_{kk}dV$ from Eq. (65) and $\gamma_2 \int_V Pe_{kk}dV$ from Eq. (71) in Eq. (59), we get:

$$\begin{aligned} &\int_V 2\mu e_{ij}e_{ij}dV + \int_V \lambda_0 e_{kk}e_{kk}dV + \int_V \rho s^2 u_i^2 dV + \frac{1}{K\left(1+\sum_{r=1}^{M^0} \frac{\tau_T^r}{r!} s^r\right)} \int_V \left(\delta^0 + \right. \\ &\left. + \sum_{r=1}^{N^0} \frac{\tau_q^r}{r!} s^r\right) sT_0 \phi_i^2 dV + \int_V l_1 T^2 dV + \frac{1}{D\left(1+\sum_{r=1}^{M^0} \frac{\tau_P^r}{r!} s^r\right)} \int_V \left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_\eta^r}{r!} s^r\right) sN_i^2 dV + \quad (72) \\ &+ \int_V nP^2 dV + 2\int_V PdTdV = 0, \end{aligned}$$

where $\int_V e_{kij}\chi_{lk,l}\, e_{ij}dV = 0$. As product of alternative tensor (e_{ijk}) with second order tensor (e_{ij}) is always equal to zero.

It is evident that $\int_V(l_1T^2 + nP^2 + 2PdT)dV$ appearing in Eq. (72) is always positive because of new thermodynamics law [12]:

$$0 < d^2 < l_1 n. \quad (73)$$

Therefore Eq. (72) vanishes at $t = 0$, owing to the homogeneous initial conditions, and it must be always non-positive for $t > 0$.

Since material parameters are positive and the integrand function in Eq. (72) is a sum of squares, thus we conclude that:

$$u_i = T = e_{ij} = t_{ij} = P = 0. \quad (74)$$

That is the Laplace transforms of the difference function (38) are zeros and according to Learch's theorem [54], the inverse Laplace transform of each is unique, consequently:

$$u_i^{(1)} = u_i^{(2)}, T^{(1)} = T^{(2)}, P^{(1)} = P^{(2)}. \quad (75)$$

Hence, uniqueness theorem is proved owing to initial and boundary restrictions.

Reciprocity theorem

Let us take a modified couple stress thermoelastic diffusion under multi-phase-lags in rotating frame of reference body occupying the region V and bounded by surface A. We consider the stress t_{ij} and the strains e_{ij} are continuous having the first derivatives whereas the displacement u_i, temperature T and chemical potential P are continuous holding continuous derivatives up to the second order, for $\boldsymbol{x} \in V + A$, $t > 0$.

Incorporating Eq. (2) in Eq. (1), yields:

$$\begin{aligned} &\left[2\mu e_{ij} - \alpha e_{kij}\chi_{lk,l} + \delta_{ij}(\lambda_0 e_{kk} - \gamma_1 T - \gamma_2 P)\right]_{,j} + \rho F_i = \rho \ddot{u}_i + \rho\left(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u})\right)_i + \\ &+ \rho(2\boldsymbol{\Omega} \times \dot{\boldsymbol{u}})_i. \end{aligned} \quad (76)$$

Equation (4) with the aid of Eqs. (5) and (6) take form:

$$\left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_q^r}{r!}\frac{\partial^r}{\partial t^r}\right)\left(l_1 T_0 \dot{T} + \gamma_1 T_0 \dot{e}_{kk} + dT_0 \dot{P} - Q\right) = K\left(1 + \sum_{r=1}^{M^0} \frac{\tau_T^r}{r!}\frac{\partial^r}{\partial t^r}\right) T_{,ii}. \quad (77)$$

Equation (7) with the aid of Eqs. (8) and (9) take form:

$$\left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_\eta^r}{r!}\frac{\partial^r}{\partial t^r}\right)\left(n\dot{P} + \gamma_2 \dot{e}_{kk} + d\dot{T} - A_0\right) = \mathrm{D}\left(1 + \sum_{r=1}^{M^0} \frac{\tau_P^r}{r!}\frac{\partial^r}{\partial t^r}\right) P_{,ii}. \quad (78)$$

We assume that the system of Eqs. (76)–(78) is assist with the boundary condition given by Eq. (13) and homogeneous initial conditions (40)–(43).

To procure dynamic reciprocity relationship for MCMRR owing to the action of body forces $F_i(\boldsymbol{x}, t)$, together with boundary condition (13), the homogeneous initial condition

(40)–(43) surface traction $h_i(\boldsymbol{x},t)$, chemical potential $\omega(\boldsymbol{x},t)$ and heating of the surface to the temperature $\upsilon(\boldsymbol{x},t)$.

Exploring Laplace transform defined by Eq. (44) on Eqs. (2), (76)–(78) and (13):

$$t_{ij,j}+\rho F_i=\rho s^2 u_i+\rho(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u}))_i+\rho(2\boldsymbol{\Omega}\times s\boldsymbol{u})_i, \tag{79}$$

$$\left[2\mu e_{ij}-\alpha e_{kij}\chi_{lk,l}+\delta_{ij}(\lambda_0 e_{kk}-\gamma_1 T-\gamma_2 P)\right]_{,j}+\rho F_i=\rho s^2 u_i+\rho\big(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u})\big)_i+ \\ +\rho(2\boldsymbol{\Omega}\times s\boldsymbol{u})_i, \tag{80}$$

$$\left(\delta^0+\sum_{r=1}^{N^0}\frac{\tau_q^r}{r!}s^r\right)(l_1 T_0 sT+\gamma_1 T_0 s e_{kk}+dT_0 sP-Q)=K\left(1+\sum_{r=1}^{M^0}\frac{\tau_T^r}{r!}s^r\right)T_{,ii}, \tag{81}$$

$$\left(\delta^0+\sum_{r=1}^{N^0}\frac{\tau_\eta^r}{r!}s^r\right)(nsP+\gamma_2 s e_{kk}+dTs-A_0)=\mathrm{D}\left(1+\sum_{r=1}^{M^0}\frac{\tau_P^r}{r!}s^r\right)P_{,ii}, \tag{82}$$

$$t_{ij}n_j=h_i(\boldsymbol{x},s), T(\boldsymbol{x},s)=\upsilon(\boldsymbol{x},s), P(\boldsymbol{x},s)=\omega(\boldsymbol{x},s),\ \boldsymbol{x}\in A\,. \tag{83}$$

We know examine two problems where applied body force, surface temperature and chemical potential are specified separately. Let the variable appear in these two problems be characterized by superscript in parentheses. Thus, we take the first problem $u_i^{(1)}, e_{ij}^{(1)}, t_{ij}^{(1)}, T^{(1)}, P^{(1)}, \eta_i^{(1)}$,... whereas for the second problem $u_i^{(2)}, e_{ij}^{(2)}, t_{ij}^{(2)}, T^{(2)}, P^{(2)}, \eta_i^{(2)}$,... . Each set of variables satisfies the system of Eqs. (55) and (79)–(83).

Let us consider the integral:

$$\int_V t_{ij}^{(1)}e_{ij}^{(2)}dV=\int_V t_{ij}^{(1)}u_{i,j}^{(2)}dV=\int_A t_{ij}^{(1)}u_i^{(2)}dA-\int_V t_{ij,j}^{(1)}u_i^{(2)}dV. \tag{84}$$

Using divergence theorem, symmetry of t_{ij} and incorporating Eqs. (79) and (83) in Eq. (84), determine:

$$\int_V t_{ij}^{(1)}e_{ij}^{(2)}dV=\int_A h_i^{(1)}u_i^{(2)}dA-\rho\int_V s^2u_i^{(1)}u_i^{(2)}dV+\rho\int_V F_i^{(1)}u_i^{(2)}dV- \\ -\rho\int_V[\big(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u})\big)_i^{(1)}+(2\boldsymbol{\Omega}\times s\boldsymbol{u})_i^{(1)}]u_i^{(2)}dV\,. \tag{85}$$

An analogous to Eq. (85), the integral $\int_V t_{ij}^{(2)}e_{ij}^{(1)}dV$ can be written as:

$$\int_V t_{ij}^{(2)}e_{ij}^{(1)}dV=\int_A h_i^{(2)}u_i^{(1)}dA-\rho\int_V s^2u_i^{(2)}u_i^{(1)}dV+\rho\int_V F_i^{(2)}u_i^{(1)}dV- \\ -\rho\int_V[\big(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u})\big)_i^{(2)}+(2\boldsymbol{\Omega}\times s\boldsymbol{u})_i^{(2)}]u_i^{(1)}dV\,. \tag{86}$$

Subtracting Eqs. (85) and (86), we get:

$$\int_V\left(t_{ij}^{(1)}e_{ij}^{(2)}-t_{ij}^{(2)}e_{ij}^{(1)}\right)dV=\int_A\left(h_i^{(1)}u_i^{(2)}-h_i^{(2)}u_i^{(1)}\right)dA+\rho\int_V\Big(F_i^{(1)}u_i^{(2)}- \\ -F_i^{(2)}u_i^{(1)}\Big)dV-\rho\int_V[\big(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u})\big)_i^{(1)}u_i^{(2)}-\big(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u})\big)_i^{(2)}u_i^{(1)}]dV- \\ -\rho\int_V[(2\boldsymbol{\Omega}\times s\boldsymbol{u})_i^{(1)}u_i^{(2)}-(2\boldsymbol{\Omega}\times s\boldsymbol{u})_i^{(2)}u_i^{(1)}]dV. \tag{87}$$

Now multiplying Eq. (45) by $e_{ij}^{(2)}$ and $e_{ij}^{(1)}$ for the first and second problem respectively, subtracting and integrating over the region V, gives:

$$\int_V\left(t_{ij}^{(1)}e_{ij}^{(2)}-t_{ij}^{(2)}e_{ij}^{(1)}\right)dV=\int_V[\gamma_1\big(T^{(2)}e^{(1)}-T^{(1)}e^{(2)}\big)-\gamma_2\big(P^{(1)}e^{(2)}-P^{(2)}e^{(1)}\big)]dV, \tag{88}$$

where $e_{ij}\delta_{ij}=e_{kk}=e$.

Using Eq. (87) in Eq. (88), yields:

$$\int_A\left(h_i^{(1)}u_i^{(2)}-h_i^{(2)}u_i^{(1)}\right)dA+\rho\int_V\left(F_i^{(1)}u_i^{(2)}-F_i^{(2)}u_i^{(1)}\right)dV- \\ -\rho\int_V[\big(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u})\big)_i^{(1)}u_i^{(2)}-\big(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u})\big)_i^{(2)}u_i^{(1)}]dV-\rho\int_V\Big[(2\boldsymbol{\Omega}\times s\boldsymbol{u})_i^{(1)}u_i^{(2)}- \\ -(2\boldsymbol{\Omega}\times s\boldsymbol{u})_i^{(2)}u_i^{(1)}\Big]\mathrm{d}V=\int_V[\gamma_1\big(T^{(2)}e^{(1)}-T^{(1)}e^{(2)}\big)-\gamma_2\big(P^{(1)}e^{(2)}-P^{(2)}e^{(1)}\big)]dV. \tag{89}$$

To derive second part, multiplying (81) by $T^{(2)}$ and $T^{(1)}$ for the first and second problem respectively, subtract and integrate over the region V:

$$\left(\delta^0+\sum_{r=1}^{N^0}\frac{\tau_q^r}{r!}s^r\right)[\gamma_1 sT_0\int_V(e^{(1)}T^{(2)}-e^{(2)}T^{(1)})dV+dT_0 s\int_V(P^{(1)}T^{(2)}-P^{(2)}T^{(1)})dV- \\ -\int_V(Q^{(1)}T^{(2)}-Q^{(2)}T^{(1)})dV]=K\left(1+\sum_{r=1}^{M^0}\frac{\tau_T^r}{r!}s^r\right)\int_V\left(T_{,ii}^{(1)}T^{(2)}-T_{,ii}^{(2)}T^{(1)}\right)dV. \tag{90}$$

Using divergence theorem and with the aid of Eq. (83), Eq. (90) takes the form:

$$\begin{aligned}&\left(\delta^0+\sum_{r=1}^{N^0}\frac{\tau_q^r}{r!}s^r\right)[\gamma_1 sT_0\int_V(e^{(1)}T^{(2)}-e^{(2)}T^{(1)})+dV\,dT_0 s\int_V(P^{(1)}T^{(2)}-P^{(2)}T^{(1)})dV-\\&-\int_V(Q^{(1)}T^{(2)}-Q^{(2)}T^{(1)})dV]=K\left(1+\sum_{r=1}^{M^0}\frac{\tau_T^r}{r!}s^r\right)\int_A(T_{,i}^{(1)}v^{(2)}-T_{,i}^{(2)}v^{(1)})dA.\end{aligned}\tag{91}$$

Equation (91) established the second part of the reciprocity theorem which involves the thermal causes of the motion v: heating of the surface A. To derive third part, multiply Eq. (82) by $P^{(2)}$ and $P^{(1)}$ to the first and second problem respectively, then integrate over the volume V.

$$\begin{aligned}&\left(\delta^0+\sum_{r=1}^{N^0}\frac{\tau_\eta^r}{r!}s^r\right)[\gamma_2 s\int_V\left(e^{(1)}P^{(2)}-e^{(2)}P^{(1)}\right)dV+ds\int_V\left(T^{(1)}P^{(2)}-T^{(2)}P^{(1)}\right)dV-\\&-\int_V\left(A_0^{(1)}P^{(2)}-A_0^{(2)}P^{(1)}\right)dV]=\left(1+\sum_{r=1}^{M^0}\frac{\tau_P^r}{r!}s^r\right)\int_V\left(P_{,ii}^{(1)}P^{(2)}-P_{,ii}^{(2)}P^{(1)}\right)dV.\end{aligned}\tag{92}$$

Using divergence theorem and with the aid of Eq. (83), Eq. (92) takes form:

$$\begin{aligned}&\left(\delta^0+\sum_{r=1}^{N^0}\frac{\tau_\eta^r}{r!}s^r\right)[\gamma_2 s\int_V\left(e^{(1)}P^{(2)}-e^{(2)}P^{(1)}\right)dV+sd\int_V\left(T^{(1)}P^{(2)}-T^{(2)}P^{(1)}\right)dV-\\&-\int_V\left(A_0^{(1)}P^{(2)}-A_0^{(2)}P^{(1)}\right)dV]=D\left(1+\sum_{r=1}^{M^0}\frac{\tau_P^r}{r!}s^r\right)\int_A\left(P_{,i}^{(1)}\omega^{(2)}-P_{,i}^{(2)}\omega^{(1)}\right)dA.\end{aligned}\tag{93}$$

Equation (93) constitutes the third part of reciprocity theorem with containing the chemical potential causes of motion. Using the integral $\int_V(e^{(1)}T^{(2)}-e^{(2)}T^{(1)})dV$ from Eq. (91) and $\int_V\left(e^{(1)}P^{(2)}-e^{(2)}P^{(1)}\right)dV$ from Eq. (93) in Eq. (89), we obtain:

$$\begin{aligned}&K\left(1+\sum_{r=1}^{M^0}\frac{\tau_T^r}{r!}s^r\right)R_2\int_A(T_{,i}^{(1)}v^{(2)}-T_{,i}^{(2)}v^{(1)})dA+T_0R_1D\left(1++\sum_{r=1}^{M^0}\frac{\tau_P^r}{r!}s^r\right)\times\\&\times\int_A\left(P_{,i}^{(1)}\omega^{(2)}-P_{,i}^{(2)}\omega^{(1)}\right)dA-R_1sT_0R_2\int_A\left(h_i^{(1)}u_i^{(2)}-h_i^{(2)}u_i^{(1)}\right)dA-\\&-\rho sT_0R_1R_2\int_V\left(F_i^{(1)}u_i^{(2)}-F_i^{(2)}u_i^{(1)}\right)dV+\rho sT_0R_1R_2\times\int_V[(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u}))_i^{(1)}u_i^{(2)}-\\&-(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u}))_i^{(2)}u_i^{(1)}]dV\rho sT_0R_1R_2\int_V\left[(2\boldsymbol{\Omega}\times s\boldsymbol{u})_i^{(1)}u_i^{(2)}-(2\boldsymbol{\Omega}\times s\boldsymbol{u})_i^{(2)}u_i^{(1)}\right]dV+\\&+R_1R_2\int_V(Q^{(1)}T^{(2)}-Q^{(2)}T^{(1)})dV+T_0R_1R_2\int_V\left(A_0^{(1)}P^{(2)}--A_0^{(2)}P^{(1)}\right)dV=0,\end{aligned}\tag{94}$$

where, $\left(\delta^0+\sum_{r=1}^{N^0}\frac{\tau_q^r}{r!}s^r\right)=R_1$, $\left(\delta^0+\sum_{r=1}^{N^0}\frac{\tau_\eta^r}{r!}s^r\right)=R_2$. Equation (94) represents the reciprocity theorem (general form) in the transformed domain.

For the implementation of the inverse Laplace transform on Eqs. (89), (91), (93) and (94), we shall use the following convolution theorem and symbolic notations:

$$L^{-1}\{F(s)G(s)\}=\int_0^t f(t-\zeta)g(\zeta)d\zeta=\int_0^t g(t-\zeta)f(\zeta)d\zeta,\tag{95}$$

$$\Lambda_1\big(f(\boldsymbol{x},\zeta)\big)=\left(\delta^0 f(\boldsymbol{x},\zeta)+\sum_{r=1}^{N^0}\frac{\tau_q^r}{r!}\frac{\partial^r f(\boldsymbol{x},\zeta)}{\partial\zeta^r}\right),\tag{96}$$

$$\Lambda_2\big(f(\boldsymbol{x},\zeta)\big)=\left(\delta^0 f(\boldsymbol{x},\zeta)+\sum_{r=1}^{N^0}\frac{\tau_\eta^r}{r!}\frac{\partial^r f(\boldsymbol{x},\zeta)}{\partial\zeta^r}\right),\tag{97}$$

$$\Lambda_3\big(f(\boldsymbol{x},\zeta)\big)=\left(\delta^0+\sum_{r=1}^{N^0}\frac{\tau_q^r}{r!}\frac{\partial^r}{\partial\zeta^r}\right)\left(\delta^0 f(\boldsymbol{x},\zeta)+\sum_{r=1}^{N^0}\frac{\tau_\eta^r}{r!}\frac{\partial^r f(\boldsymbol{x},\zeta)}{\partial\zeta^r}\right).\tag{98}$$

Thus, we get the first, second, third and general forms of the reciprocity theorem in the resulting form:

$$\begin{aligned}&\int_A\int_0^t h_i^{(1)}(\boldsymbol{x},t-\zeta)u_i^{(2)}(\boldsymbol{x},\zeta)d\zeta dA+\rho\int_V\int_0^t F_i^{(1)}(\boldsymbol{x},t-\zeta)u_i^{(2)}(\boldsymbol{x},\zeta)d\zeta dV+\\&+\gamma_1\int_V\int_0^t T^{(1)}(\boldsymbol{x},t-\zeta)e^{(2)}(\boldsymbol{x},\zeta)d\zeta dV+\gamma_2\int_V\int_0^t P^{(1)}(\boldsymbol{x},t-\zeta)e^{(2)}(\boldsymbol{x},\zeta)d\zeta dV-\\&-\rho\int_V\int_0^t(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u}))_i^{(1)}(\boldsymbol{x},t-\zeta)u_i^{(2)}(\boldsymbol{x},\zeta)d\zeta dV-\\&-\rho\int_V\int_0^t\left(2\boldsymbol{\Omega}\times\frac{\partial}{\partial\zeta}\boldsymbol{u}\right)_i^{(1)}(\boldsymbol{x},t-\zeta)u_i^{(2)}(\boldsymbol{x},\zeta)d\zeta dV=S_{21}^{12},\end{aligned}\tag{99}$$

$$\begin{aligned}&K\int_A\int_0^t T_{,i}^{(1)}(\boldsymbol{x},t-\zeta)v^{(2)}(\boldsymbol{x},\zeta)d\zeta dA+\\&+K\sum_{r=1}^{M^0}\frac{\tau_T^r}{r!}\int_A\int_0^t T_{,i}^{(1)}(\boldsymbol{x},t-\zeta)\frac{\partial^r}{\partial\zeta^r}v^{(2)}(\boldsymbol{x},\zeta)d\zeta dA-\\&-\gamma_1T_0\int_V\int_0^t e^{(1)}(\boldsymbol{x},t-\zeta)\frac{\partial}{\partial\zeta}\Lambda_1\left(T^{(2)}(\boldsymbol{x},\zeta)\right)d\zeta dV-dT_0\int_V\int_0^t P^{(1)}(\boldsymbol{x},t-\zeta)\times\end{aligned}\tag{100}$$

$$\times \frac{\partial}{\partial \zeta} \Lambda_1 \left(T^{(2)}(\boldsymbol{x},\zeta)\right) d\zeta dV + \int_V \int_0^t Q^{(1)}(\boldsymbol{x},t-\zeta) \Lambda_1 \left(T^{(2)}(\boldsymbol{x},\zeta)\right) d\zeta dV = S_{21}^{12},$$

$$\begin{aligned}
&D \int_A \int_0^t P_{,i}^{(1)}(\boldsymbol{x},t-\zeta)\omega^{(2)}(\boldsymbol{x},\zeta) d\zeta dA + \\
&+D \sum_{r=1}^{M^0} \frac{\tau_P^r}{r!} \int_A \int_0^t P_{,i}^{(1)}(\boldsymbol{x},t-\zeta) \frac{\partial^r}{\partial \zeta^r} \omega^{(2)}(\boldsymbol{x},\zeta) d\zeta dA - \\
&-\gamma_2 \int_V \int_0^t e^{(1)}(\boldsymbol{x},t-\zeta) \frac{\partial}{\partial \zeta} \Lambda_2 \left(P^{(2)}(\boldsymbol{x},\zeta)\right) d\zeta dV - \\
&-d \int_V \int_0^t T^{(1)}(\boldsymbol{x},t-\zeta) \times \frac{\partial}{\partial \zeta} \Lambda_2 \left(P^{(2)}(\boldsymbol{x},\zeta)\right) d\zeta dV \\
&+ \int_V \int_0^t A_0^{(1)}(\boldsymbol{x},t-\zeta) \Lambda_2 \left(P^{(2)}(\boldsymbol{x},\zeta)\right) d\zeta dV = S_{21}^{12},
\end{aligned} \tag{101}$$

$$\begin{aligned}
&K \int_A \int_0^t T_{,i}^{(1)}(\boldsymbol{x},t-\zeta) \Lambda_2 \left(v^{(2)}(\boldsymbol{x},\zeta)\right) d\zeta dA + \\
&+K \sum_{r=1}^{M^0} \frac{\tau_T^r}{r!} \int_A \int_0^t T_{,i}^{(1)}(\boldsymbol{x},t-\zeta) \frac{\partial^r}{\partial \zeta^r} \Lambda_2 \left(v^{(2)}(\boldsymbol{x},\zeta)\right) d\zeta dA + \\
&+DT_0 \int_A \int_0^t P_{,i}^{(1)}(\boldsymbol{x},t-\zeta) \Lambda_1 \left(\omega^{(2)}(\boldsymbol{x},\zeta)\right) d\zeta dA + \\
&+T_0 D \sum_{r=1}^{M^0} \frac{\tau_P^r}{r!} \int_A \int_0^t P_{,i}^{(1)}(\boldsymbol{x},t-\zeta) \frac{\partial^r}{\partial \zeta^r} \Lambda_1 \left(\omega^{(2)}(\boldsymbol{x},\zeta)\right) d\zeta dA - \\
&-\rho T_0 \int_V \int_0^t F_i^{(1)}(\boldsymbol{x},t-\zeta) \frac{\partial}{\partial \zeta} \Lambda_3 \left(u_i^{(2)}(\boldsymbol{x},\zeta)\right) d\zeta dV - \\
&-T_0 \int_A \int_0^t h_i^{(1)}(\boldsymbol{x},t-\zeta) \times \frac{\partial}{\partial \zeta} \Lambda_3 \left(u_i^{(2)}(\boldsymbol{x},\zeta)\right) d\zeta dA + \\
&+\rho T_0 \int_V \int_0^t \left(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u})\right)_i^{(1)}(\boldsymbol{x},t-\zeta) \frac{\partial}{\partial \zeta} \Lambda_3 \left(u_i^{(2)}(\boldsymbol{x},\zeta)\right) d\zeta dV + \\
&+\rho T_0 \int_V \int_0^t \left(2\boldsymbol{\Omega} \times \frac{\partial}{\partial \zeta} \boldsymbol{u}\right)_i^{(1)} (\boldsymbol{x},t-\zeta) \frac{\partial}{\partial \zeta} \Lambda_3 \left(u_i^{(2)}(\boldsymbol{x},\zeta)\right) d\zeta dV + \\
&+ \int_V \int_0^t Q^{(1)}(\boldsymbol{x},t-\zeta) \Lambda_3 \left(T^{(2)}(\boldsymbol{x},\zeta)\right) d\zeta dV + \\
&+T_0 \int_V \int_0^t A_0^{(1)}(\boldsymbol{x},t-\zeta) \Lambda_3 \left(P^{(2)}(\boldsymbol{x},\zeta)\right) d\zeta dV = S_{21}^{12}.
\end{aligned} \tag{102}$$

Here S_{21}^{12} indicates the same expression on left hand side except that the superscripts (1) and (2) interchanged. Hence the reciprocity theorem is proved.

Variational criterion

The criterion of virtual work with variations of displacement for deformable elastic body with thermal and chemical potential is given as

$$\int_V \rho(F_i - \ddot{u}_i + (\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u}))_i + (2\boldsymbol{\Omega} \times \dot{\boldsymbol{u}})_i)\delta u_i dV + \int_A h_i \delta u_i dA = \int_V t_{ij} \delta u_{i,j} dV. \tag{103}$$

In Eq. (103) left hand side corresponds to virtual work of body forces F_i, inertial forces $\rho\ddot{u}_i$, surface force h_i=$t_{ji}n_i$ although the right hand side corresponds to the virtual work of internal forces. We denote by n_i the outward unit normal of A.

Equation (103) with the aid of symmetry of stress tensor and the definition of strain tensor can be written in another form as: substitute the value of t_{ij} from Eq. (1) in Eq. (103):

$$\begin{aligned}
&\int_V \rho \left(F_i - \ddot{u}_i + \left(\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{u})\right)_i + (2\boldsymbol{\Omega} \times \dot{\boldsymbol{u}})_i\right) \delta u_i dV + \int_A L \delta u_i dA = \delta W - \\
&- \int_V \alpha e_{kij} \chi_{lk,l} \delta u_{i,j} dV - \gamma_1 \int_V \mathrm{T} \delta e_{kk} dV - \gamma_2 \int_V P \delta e_{kk} dV,
\end{aligned} \tag{104}$$

$$W = \mu \int_V e_{ij} e_{ij} dV + \frac{\lambda_0}{2} \int_V e_{kk} e_{kk} dV, \tag{105}$$

where W is strain density function.

Taking into consideration (60) and introducing heat source term Q_1 such that Q=$T_0 \dot{Q}_1$. Then, the Eqs. (4) and (5) with the aid of (6) reduce to:

$$\delta \phi_{i,i} = -l_1 \delta T - \gamma_1 \delta e_{kk} - d \delta P + \delta Q_1, \tag{106}$$

$$\frac{T_0}{K}J_i + T_{,i} = 0, \tag{107}$$

where

$$J_i = \left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_q^r}{r!}\frac{\partial^r}{\partial t^r}\right)\dot{\phi}_i + \frac{K}{T_0}\sum_{r=1}^{M^0} \frac{\tau_T^r}{r!}\frac{\partial^r}{\partial t^r}T_{,i}. \tag{108}$$

Multiplying the Eq. (107) by $\delta\phi_i$ and integrating over the volume V, yields:

$$\int_V \left(\frac{T_0}{K}J_i + T_{,i}\right)\delta\phi_i dV = 0\,. \tag{109}$$

Equation (109) with the aid of the relation $T_{,i}\delta N_i = (T\delta N_i)_{,i} - T\delta N_{i,i}$ and Gauss's divergence theorem can be written as:

$$\int_A T\delta\phi_i n_i dA - \int_V T\delta\phi_{i,i} dV + \frac{T_0}{K}\int_V J_i\delta\phi_i dV = 0. \tag{110}$$

Using value of $\delta\phi_{i,i}$ in Eq. (110), we obtain:

$$\int_A T\delta\phi_i n_i dA + \delta P_1 + \gamma_1\int_V T\delta e_{kk} dV + d\int_V T\delta P dV - \int_V T\delta Q_1 dV + \delta D_1 = 0, \tag{111}$$

where $\delta P_1 = l_1\int_V T\delta T dV$, $\delta D_1 = \frac{T_0}{K}\int_V J_i\delta\phi_i dV$.

Taking into consideration (66) and introducing chemical potential source term M_2 such that $A_0 = \dot{M}_2$. Then, Eqs. (7) and (8) with the aid of Eq. (9) reduce to:

$$\delta N_{i,i} = -n\delta P - \gamma_2\delta e_{kk} - d\delta T + \delta M_2, \tag{112}$$

$$\frac{Y_i}{D} + P_{,i} = 0, \tag{113}$$

where $Y_i = \left(\delta^0 + \sum_{r=1}^{N^0} \frac{\tau_\eta^r}{r!}\frac{\partial^r}{\partial t^r}\right)\dot{N}_i + D\sum_{r=1}^{M^0} \frac{\tau_P^r}{r!}\frac{\partial^r}{\partial t^r}P_{,i}$

Multiplying the Eq. (113) by δN_i and integrating over the volume V, yields:

$$\int_V \left(\frac{Y_i}{D} + P_{,i}\right)\delta N_i dV = 0. \tag{114}$$

Equation (114) with the aid of the relation $P_{,i}\delta N_i = (P\delta N_i)_{,i} - P\delta N_{i,i}$ and Gauss's divergence theorem can be written as:

$$\int_V \frac{Y_i}{D}\delta N_i dV + \int_A P\delta N_i n_i dA - \int_V P\delta N_{i,i} dV = 0. \tag{115}$$

Using value of $\delta N_{i,i}$ from Eq. (112) in Eq. (115):

$$\delta D_2 + \delta P_2 + \int_A P\delta N_i n_i dA + \int_V \gamma_2 P\delta e_{kk} dV + \int_V Pd\delta T dV - \int_V P\delta M_2 dV = 0. \tag{116}$$

where $D_2 = \frac{1}{D}\int_V Y_i\,\delta N_i dV$, $\delta P_2 = \int_V nP\delta P dV$.

Equation (104) with the aid of Eq. (111), and Eq. (116) can be written as:

$$\begin{aligned}&\delta\left(W + D_1 + P_1 + D_2 + P_2 + d\int_V PT dV\right) = \int_V \rho\Big(F_i - \ddot{u}_i + \left(\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{u})\right)_i + \\ &+ (2\boldsymbol{\Omega}\dot{\boldsymbol{u}})_i\Big)\delta u_i dV + \int_A L\delta u_i dA + \int_V \alpha e_{kij}\chi_{lk,l}\delta u_{i,j} dV - \int_A T\delta\phi_i n_i dA - \int_A P\delta N_i n_i dA + \\ &+ \int_V T\delta Q_1 dV + \int_V P\delta M_2 dV.\end{aligned} \tag{117}$$

Equation (117) represents variational principle for thermoelastic diffusion body under multi-phase-lags. In Eq. (117), right hand side interpreted as the work done by the body force, surface traction, inertia forces, heating of the surface, heat source and mass diffusion source with the virtual deformation, whereas left hand side explores the variation of the sum of the dissipation function, heat potential, chemical potential and the work of deformation.

Applications

Consider an infinite isotropic thermoelastic medium in absence of rotation and couple stress parameters in which the surface integrals are removed in Eq. (94), the body forces, heat sources and chemical potential sources are acting in bounded region only.

Also considering $C_r = \{F_i^{(r)}(\boldsymbol{x},t), h_i^{(r)}(\boldsymbol{x},t), U_i^{(r)}(\boldsymbol{x},t), v^{(r)}(\boldsymbol{x},t), Q^{(r)}(\boldsymbol{x},t), \omega^{(r)}(\boldsymbol{x},t), A_0^{(r)}(\boldsymbol{x},t)\}$ and thus, we get relation:

$$\begin{aligned}&s\rho T_0 \int_V \left(F_i^{(1)}u_i^{(2)} - F_i^{(2)}u_i^{(1)}\right) dV - \int_V \left(Q^{(1)}T^{(2)} - Q^{(2)}T^{(1)}\right) dV - \\ &-T_0 \int_V \left(A_0^{(1)}P^{(2)} - A_0^{(2)}P^{(1)}\right) dV = 0.\end{aligned} \tag{118}$$

For implementation of the inverse Laplace transform defined by Eq. (44) in (118), determine:

$$\begin{aligned}&\rho T_0 \int_V \int_0^t F_i^{(1)}(\boldsymbol{x}, t-\zeta) \frac{\partial}{\partial \zeta} u_i^{(2)}(\boldsymbol{x}, \zeta) d\zeta dV - \int_V \int_0^t Q^{(1)}(\boldsymbol{x}, t-\zeta) T^{(2)}(\boldsymbol{x}, \zeta) d\zeta - \\ &-T_0 \int_V \int_0^t A_0^{(1)}(\boldsymbol{x}, t-\zeta) P^{(2)}(\boldsymbol{x}, \zeta) d\zeta dV = S_{21}^{12}.\end{aligned} \tag{119}$$

Here S_{21}^{12} demonstrates the similar expression on left hand side barring the superscripts (1) and (2) can be reciprocated.

Assuming the point η_1 and η_2, the instantaneous concentrated body forces $F_i^{(1)}$ and $F_i^{(2)}$acting in the direction of $\boldsymbol{x}_j$-axis and $\boldsymbol{x}_k$-axis, respectively. Thus, we obtain:

$$\begin{aligned}&C_1 = \{F_i^{(1)}, 0, 0, 0, 0, 0, 0\},\ C_2 = \{F_i^{(2)}, 0, 0, 0, 0, 0, 0\},\ F_i^{(1)} = \delta(\boldsymbol{x} - \eta_1)\delta_{ij}\delta(t), \\ &F_i^{(2)} = \delta(\boldsymbol{x} - \eta_2)\delta_{ij}\delta(t),\ u_i^{(1)} \equiv u_i^{(1)}(\boldsymbol{x}, \eta_1, t),\ u_i^{(2)} \equiv u_i^{(2)}(\boldsymbol{x}, \eta_2, t).\end{aligned} \tag{120}$$

Incorporating Eq. (120) in (119), yield:

$$\dot{u}_k^{(1)}(\eta_2, \eta_1, t) = \dot{u}_j^{(2)}(\eta_1, \eta_2, t). \tag{121}$$

It means that the velocity of the displacement $u_j^{(2)}$ at the point η_1 due to the action of instantaneous concentrated forces acting at point η_2 in the direction $\boldsymbol{x}_k$-axis is equal to the velocity of the displacement $u_k^{(1)}$ at the point η_2 produced by the action of a concentrated forces acting at the point η_1 in the direction of the $\boldsymbol{x}_j$-axis.

Presuming the point η_1 and η_2, there act instantaneous concentrated heat sources $Q^{(1)}$ and $Q^{(2)}$. Thus, we obtain:

$$\begin{aligned}&C_1 = \{0, 0, 0, 0, Q^{(1)}, 0, 0\},\ C_2 = \{0, 0, 0, 0, Q^{(2)}, 0, 0\},\ Q^{(1)} = \delta(\boldsymbol{x} - \eta_1)\delta(t), \\ &Q^{(2)} = \delta(\boldsymbol{x} - \eta_2)\delta(t),\ T^{(1)} \equiv T^{(1)}(\boldsymbol{x}, \eta_1, t),\ T^{(2)} \equiv T^{(2)}(\boldsymbol{x}, \eta_2, t).\end{aligned} \tag{122}$$

Incorporating Eq. (122) in (119), yield:

$$T^{(1)}(\eta_2, \eta_1, t) = T^{(2)}(\eta_1, \eta_2, t). \tag{123}$$

It is concluded that the temperature $T^{(2)}$ at the point η_1 due to the action of heat sources at point η_2 is equal to the temperature $T^{(1)}$ at the point η_2 produced by the action of heat sources atthe point η_1.

Considering the point η_1 and η_2, there act instantaneous concentrated chemical potential sources $A_0^{(1)}$ and $A_0^{(2)}$. Thus, we obtain:

$$\begin{aligned}&C_1 = \{0, 0, 0, 0, 0, 0, A_0^{(1)}\},\ C_2 = \{0, 0, 0, 0, 0, 0, A_0^{(2)}\},\ A_0^{(1)} = \delta(\boldsymbol{x} - \eta_1)\delta(t), \\ &A_0^{(2)} = \delta(\boldsymbol{x} - \eta_2)\delta(t),\ P^{(1)} \equiv P^{(1)}(\boldsymbol{x}, \eta_1, t),\ P^{(2)} \equiv P^{(2)}(\boldsymbol{x}, \eta_2, t).\end{aligned} \tag{124}$$

Incorporating Eq. (124) in (119), yield:

$$P^{(1)}(\eta_2, \eta_1, t) = P^{(2)}(\eta_1, \eta_2, t). \tag{125}$$

Thus, the chemical potential $P^{(2)}$ at the point η_1 due to the action of chemical potential sources at point η_2 is equal to the chemical potential $P^{(1)}$ at the point η_2 produced by the action of chemical potential sources at the point η_1.

Taking the point η_1 and η_2, there act instantaneous concentrated body forces $F_i^{(1)}$ acts in the direction $\boldsymbol{x}_j$-axis and heat sources $Q^{(2)}$, respectively. Thus, we obtain:

$$\begin{aligned}&C_1 = \{F_i^{(1)}, 0, 0, 0, 0, 0, 0\},\ C_2 = \{0, 0, 0, 0, Q^{(2)}, 0, 0\},\ F_i^{(1)} = \delta(\boldsymbol{x} - \eta_1)\delta_{ij}\delta(t), \\ &Q^{(2)} = \delta(\boldsymbol{x} - \eta_2)\delta(t),\ F_i^{(2)} = 0;\ Q^{(1)} = 0,\ u_i^{(1)} \equiv u_i^{(1)}(\boldsymbol{x}, \eta_1, t),\ T^{(2)} \equiv T^{(2)}(\boldsymbol{x}, \eta_2, t).\end{aligned} \tag{126}$$

Incorporating Eq. (126) in (119), yield:

$$T^{(1)}(\eta_2,\eta_1,t) = -\rho T_0 \dot{u}_j^{(2)}(\eta_1,\eta_2,t), \tag{127}$$

where $T^{(1)}(\eta_2,\eta_1,t)$ is the temperature at η_2 due to $F_i^{(1)}$ acting at η_1 in the direction $\boldsymbol{x}_j$-axis and $\dot{u}_j^{(2)}(\eta_1,\eta_2,t)$ is the velocity of particle at η_1 due to $Q^{(2)}$ located at η_2.

Imaging the point η_1 and η_2, there act instantaneous concentrated body forces $F_i^{(1)}$ acts in the direction $\boldsymbol{x}_j$-axis and chemical potential sources $A_0^{(2)}$, respectively. Thus, we obtain:

$$\begin{gathered} C_1 = \{F_i^{(1)},0,0,0,0,0,0\},\ C_2 = \{0,0,0,0,0,0,A_0^{(2)}\},\ F_i^{(1)} = \delta(\boldsymbol{x}-\eta_1)\delta_{ij}\delta(t), \\ A_0^{(2)} = \delta(\boldsymbol{x}-\eta_2)\delta(t),\ F_i^{(2)} = 0,\ A_0^{(1)} = 0,\ u_i^{(1)} \equiv u_i^{(1)}(\boldsymbol{x},\eta_1,t),\ P^{(2)} \equiv P^{(2)}(\boldsymbol{x},\eta_2,t). \end{gathered} \tag{128}$$

Incorporating Eq. (128) in (119), yield:

$$P^{(1)}(\eta_2,\eta_1,t) = -\rho \dot{u}_j^{(2)}(\eta_1,\eta_2,t), \tag{129}$$

where $P^{(1)}(\eta_2,\eta_1,t)$ is the chemical potential at η_2 due to $F_i^{(1)}$ acting at η_1 in the direction $\boldsymbol{x}_j$-axis and $\dot{u}_j^{(2)}(\eta_1,\eta_2,t)$ is the velocity of particle at η_1 due to $A_0^{(2)}$ located at η_2.

Considering the point η_1 and η_2, there act instantaneous concentrated heat sources $Q^{(1)}$ and instantaneous chemical potential $A_0^{(2)}$. Thus, we get:

$$\begin{gathered} C_1 = \{0,0,0,0,\ Q^{(1)},0,0\};\ C_2 = \{0,0,0,0,0,0,A_0^{(2)}\};\ Q^{(1)} = \delta(\boldsymbol{x}-\eta_1)\ \delta(t); \\ A_0^{(2)} = \delta(\boldsymbol{x}-\ \eta_2)\ \delta(t);\ P^{(1)} \equiv P^{(1)}(\boldsymbol{x},\eta_1,t);\ T^{(2)} \equiv T^{(2)}(\boldsymbol{x},\eta_2,t);\ Q^{(2)} = A_0^{(1)} = 0. \end{gathered} \tag{130}$$

Incorporating Eq. (130) in (119), yield:

$$T_0 P^{(1)}(\eta_2,\eta_1,t) = T^{(2)}(\eta_1,\eta_2,t). \tag{131}$$

It is concluded that the temperature $T^{(2)}$ at the point η_1 due to the action of chemical sources at point η_2 is equal to the chemical potential $P^{(1)}$ at the point η_2 produced by the action of heat sources at the point η_1.

Consider heat source $Q^{(1)}$ moving with uniform speed v in an infinite thermoviscoelastic medium in the direction $\boldsymbol{x}_3$ and an instantaneous concentrated heat source $Q^{(2)}$ located at η_2, thus we have:

$$\begin{gathered} C_1 = \{0,0,0,0,Q^{(1)},0,0\},\ C_2 = \{0,0,0,0,Q^{(2)},0,0\},\ Q^{(1)} = \delta(x_1)\delta(x_2)\delta(x_3 - vt), \\ Q^{(2)} = \delta(\boldsymbol{x}-\eta_2)\delta(t),\ T^{(1)} \equiv T^{(1)}(\boldsymbol{x},t),\ T^{(2)} \equiv T^{(2)}(\boldsymbol{x},\eta_2,t). \end{gathered} \tag{132}$$

Incorporating Eq. (132) in (119), yield:

$$T^{(1)}(\eta_2,t) = \int_0^t T^{(2)}(\varsigma,\eta_2\ ,t-\tau)d\tau, \tag{133}$$

where $\varsigma = (0,0,vt)$ also the temperature field $T^{(1)}$due to the moving heat source $Q^{(1)}$ in the terms of the temperature $T^{(2)}$ due to the instantaneous concentrated heat source $Q^{(2)}$.

To evaluate the velocity field due to moving heat source $Q^{(1)}$ and the instantaneous concentrated body force $F_i^{(2)}$acting at point η_2 in the direction of $\boldsymbol{x}_j$-axis, consider:

$$\begin{gathered} C_1 = \{0,0,0,0,Q^{(1)},0,0\},\ C_2 = \{F_i^{(2)},0,0,0,0,0,0\},\ Q^{(1)} = \delta(x_1)\delta(x_2)\delta(x_3 - vt), \\ F_i^{(2)} = \delta(\boldsymbol{x}-\eta_2)\delta_{ij}\delta(t),\ Q^{(2)} = F_i^{(1)}{=}0;\ u_i^{(1)} \equiv u_i^{(1)}(\boldsymbol{x},t),\ T^{(2)} \equiv T^{(2)}(\boldsymbol{x},\eta_2,t), \end{gathered} \tag{134}$$

$$\dot{u}_j^{(2)}(\eta_2,t) = -\frac{1}{\rho T_0}\int_0^t T^{(j)}\,(\varsigma,\eta_2\ ,t-\tau)d\tau, \tag{135}$$

where $T^{(j)}(\boldsymbol{x},\eta_2\ ,t)$ is the temperature at the point $\boldsymbol{x}$ due to the instantaneous concentrated body force $F_i^{(2)}$ acting at point η_2 in the direction of the x_j axis.

Consider chemical potential source $A_0^{(1)}$ moving with uniform speed v in an infinite thermoviscoelastic medium in the direction $\boldsymbol{x}_3$ and an instantaneous concentrated chemical potential source $A_0^{(2)}$ located at η_2, thus we have:

$$C_1 = \{0,0,0,0,0,0,A_0^{(1)}\},\ C_2 = \{0,0,0,0,0,0,A_0^{(2)}\},\ A_0^{(1)} = \delta(x_1)\delta(x_2)\delta(x_3 - vt), \tag{136}$$

$A_0^{(2)} = \delta(\boldsymbol{x} - \eta_2)\delta(t)$, $P^{(1)} \equiv P^{(1)}(\boldsymbol{x}, t)$, $P^{(2)} \equiv P^{(2)}(\boldsymbol{x}, \eta_2, t)$.

Incorporating Eq. (136) in (119), yield:

$$P^{(1)}(\eta_2, t) = \int_0^t P^{(2)}(\varsigma, \eta_2, t - \tau)d\tau, \tag{137}$$

where $\varsigma = (0,0, vt)$ also the chemical potential field $P^{(1)}$ due to the moving chemical potential source $A_0^{(1)}$ in the terms of the chemical potential $P^{(2)}$ due to the instantaneous concentrated chemical potential source $A_0^{(2)}$.

To evaluate the velocity field due to moving chemical potential source $A_0^{(1)}$ and the instantaneous concentrated body force $F_i^{(2)}$ acting at point η_2 in the direction of $\boldsymbol{x}_j$-axis, consider:

$$\begin{gathered} C_1 = \{0,0,0,0,0,0, A_0^{(1)}\},\ C_2 = \{F_i^{(2)}, 0,0,0,0,0,0\},\ A_0^{(1)} = \delta(x_1)\delta(x_2)\delta(x_3 - vt), \\ F_i^{(2)} = \delta(\boldsymbol{x} - \eta_2)\delta_{ij}\delta(t),\ A_0^{(2)} = F_i^{(1)} = 0,\ u_i^{(1)} \equiv u_i^{(1)}(\boldsymbol{x}, t),\ P^{(2)} \equiv P^{(2)}(\boldsymbol{x}, \eta_2, t), \end{gathered} \tag{138}$$

$$\dot{u}_j^{(2)}(\eta_2, t) = -\frac{1}{\rho}\int_0^t P^{(j)}(\varsigma, \eta_2, t - \tau)d\tau, \tag{139}$$

where $P^{(j)}(\boldsymbol{x}, \eta_2, t)$ is the chemical potential at the point $\boldsymbol{x}$ due to the instantaneous concentrated body force $F_i^{(2)}$ acting at point η_2 in the direction of the x_j axis.

Unique cases

1.1. In absence of rotation impact in Eqs. (11), (72), (102) and (117), we determine the equivalent results for modified couple stress thermoelastic diffusion under multi-phase-lags model.

1.2. Taking $\delta^0 = 1$, $\tau_T = \tau_q = \tau_\eta = \tau_P = 0$, $\alpha = 0$ and independent of rotation impact in Eqs. (11), (72), (102) and (117) yields the corresponding results for coupled thermoelastic diffusion model and these results are similar as obtained by Kumar and Gupta [25] as a special case.

1.3. Taking $\delta^0 = 1$, $\tau_T = 0$, $\tau_P = 0$, $N^0 = 1$, $\alpha = 0$ and independent of rotation impact in Eqs. (11), (72), (102) and (117) determine the resulting expressions for coupled thermoelastic diffusion under L-S model, these results are similar as obtained by Ezzat and Fayik [39] in a special case.

1.4. Letting $N^0 = 1$, $M^0 = 1$, $\delta^0 = 1$, $\alpha = 0$ and independent of rotation impact determine the resulting results for coupled thermoelastic diffusion with single-phase-lag model.

1.5. If $\delta^0 = 1$, $N^0 = 2$, $M^0 = 1$, $\alpha = 0$ and independent of rotation impact in Eqs. (11), (72), (102) and (117) yields the corresponding results for thermoelastic diffusion with dual-phase-lag model, these results are similar as obtain by Kumar and Gupta [25].

1.6. In the absence of diffusion parameter, couple stress parameter and independent of rotation impact, Eqs. (11), (72), (102) and (117) yields the results for thermoelastic with dual-phase-lag model, these results are similar as obtained by Kumar and Gupta [26] as a special case.

Conclusions

In this paper, a new establishment of modified couple stress thermoelastic diffusion under multi-phase-lags model in rotating frame of reference has been conferred. The fundamental theorems alike energy, uniqueness and reciprocity along with the variational criterion have been established in the simulated model. Laplace transform is used to prove uniqueness theorem and reciprocity theorem. With the guide of energy theorem, we can examine the motion of the system under different sources (mechanical force, thermal source and mass-diffusion source). As an application of reciprocity theorem, instantaneous concentrated body forces, heat forces and chemical potential sources along with moving heat source, chemical potential sources are taken. Uniqueness theorem is important in establishing the existence and properties of mathematical objects. With the help of reciprocity theorem, different methods of integrating differential equation of thermoelasticity using the Green function can be analyzed. The variational criterion leads in deriving the differential equation of various systems like membranes and cells easy aspect and also prepares it possible to infer distinct approximation techniques for the solution of various problems (statics and dynamics). Although the problem is theoretical, but it helps for further investigation to examine wave phenomena and variational problems in the assume model. The results obtained in particular case of the model are verified from the well-known results. The proposed model is unique in its form and physical meaning is apparent. It is convenient and provides more approaches to estimate how a material behaves in the real world. It is concluded that all the field quantities are sufficiently restricted to thermal and couple stress parameters. We observe that phase-lag and rotation have significant effect on the all the field studied and the results supporting the definition of the classification of thermal conductivity of the material. It is concluded that modified couple stress and thermoelastic diffusion fields play valuable role in processing and characterization to improve material properties. The results obtained in this study should be useful for research working in thermodynamic engineering, material science and hyperbolic thermoelastic models. The mathematical model proposed in this work is a powerful, indispensable tool for studying various problems, scientific research, product, process development and manufacturing.

CRediT authorship contribution statement

Saurav Sharma: writing-review and editing, conceptualization, investigation, supervision, data curation; **Devi Sangeeta**: writing-review and editing, writing-original draft, Investigation, data curation; **Rajneesh Kumar**: writing-original draft, conceptualization, supervision, data curation.

Conflict of interest

The authors declare that they have no conflict of interest.

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